Physics 521 Fall 2014

Problem Set #5: Due: Tuesday Oct.. 7, 2014

Problem 1: From Cohen-Tannoudji vol. I, Exercise 14, page 347. (25 points).

Consider a physical system whose state space, which is three dimensional, spanned by the orthonormal basis formed by the three kets $|u_1\rangle, |u_2\rangle, |u_3\rangle$. In this basis, the Hamiltonian operator of the system and the two observables $\hat{A}$ and $\hat{B}$ have the representations:

$$\hat{H} \doteq \hbar\omega_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \quad \hat{A} \doteq a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \quad \hat{B} \doteq b \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where ω_0, a and b are positive real constants.

The state at time t=0 is: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$.

(a) The energy of the system is measured (projectively) at t=0. What values can be found and with what probabilities? What is the mean value and rms uncertainty in energy?

(b) If one measures observable A at t=0 (projectively), what values can be found and with what probabilities? What is the state vector immediately after the measurement?

(c) Given $|\psi(0)\rangle$ and the Hamiltonian above, find $|\psi(t)\rangle$.

(d) Calculate $\langle \hat{A} \rangle(t)$ and $\langle \hat{B} \rangle(t)$, for time t>0. Comment on your result.

(e) What results can be obtained if observable A is measured (projectively) at time t? Repeat for B. Interpret.

Problem 2: Adiabatic and sudden changes in the Hamiltonian (25 points)

In many circumstances the Hamiltonian will depend *explicitly* on time through some change in a an external parameter (e.g. in Prob. 4 below, an external magnetic field is varied in time). The general solution for the time propagator is quite involved; we will revisit it next semester. There are however, two limiting cases which we can solve approximately, that are of great importance: *adiabatic* and *sudden* changes in the Hamiltonian. The goal of this problem is to quantify what these words mean, and to get a solution for the state vector in these time *dependent* systems.

Consider a Hamiltonian with explicit time dependence $\hat{H}(t)$. At every time t, $\hat{H}(t)$, being an Hermitian operator, has a complete set of orthonormal eigenvectors:

$$\hat{H}(t)\left|u_n^{(t)}\right\rangle = E_n^{(t)}\left|u_n^{(t)}\right\rangle, \quad \left\langle u_n^{(t)} \middle| u_m^{(t)}\right\rangle = \delta_{nm}.$$

Note: The eigenvectors and eigenvalues are *parameterized* by t. This is **NOT** to be confused with the time *dependent* solution to the Schrödinger equation: $\left|u_n^{(t)}\right\rangle \neq \hat{U}(t)\left|u_n^{(0)}\right\rangle$.

Suppose at time $t=0$, we have the state vector which we expand in the "initial eigenstates",

$$|\psi(0)\rangle = \sum_n c_n^{(0)}\left|u_n^{(0)}\right\rangle$$

For concreteness, let us suppose choose: $\hat{H}(t) = \hat{H}_0 + \left(\frac{t}{T}\right)\hat{H}_1$, where $\hat{H}_0$ and $\hat{H}_1$ are time independent Hermitian Hamiltonians.

The goal of this problem is to show that for:

Sudden change: $|\psi(T)\rangle = \sum_n c_n^{(0)}\left|u_n^{(0)}\right\rangle$ (up to an overall phase).

Adiabatic Change: $|\psi(T)\rangle = \sum_n a_n^{(T)}\left|u_n^{(T)}\right\rangle$, where $a_n^{(T)} = c_n^{(0)} \exp\left(-i/\hbar \int_o^T E_n^{(t')} dt'\right)$.

(a) Make the general Ansatz: $|\psi(t)\rangle = \sum_n c_n^{(t)} e^{-i\phi_n(t)}\left|u_n^{(t)}\right\rangle$, where $\phi_n(t) = \int_o^t E_n^{(t)} dt'/\hbar$ (in order words factor out the expected kind of phase evolution).

$$\textbf{Show that: } \frac{dc_m^{(t)}}{dt} = -\sum_n c_n^{(t)}\, e^{-i(\phi_n - \phi_m)}\left\langle u_m^{(t)} \middle| \frac{d}{dt}\middle| u_n^{(t)}\right\rangle.$$

(b) Now **show that** $\left\langle u_m^{(t)} \right| \frac{d}{dt} \left| u_n^{(t)} \right\rangle = \begin{cases} \dfrac{\left\langle u_m^{(t)} \right| \frac{d\hat{H}}{dt} \left| u_n^{(t)} \right\rangle}{E_n^{(t)} - E_m^{(t)}} = \dfrac{\left\langle u_m^{(t)} \right| \hat{H}_1 \left| u_n^{(t)} \right\rangle}{\hbar \omega_{nm}^{(t)} T}, & n \neq m \\ 0, & n = m. \end{cases}$

and thus **show** $\dfrac{dc_m^{(t)}}{dt} = -\sum_{n \neq m} c_n^{(t)} e^{-i(\phi_n - \phi_m)} \dfrac{\left\langle u_m^{(t)} \right| \hat{H}_1 \left| u_n^{(t)} \right\rangle}{\hbar \omega_{nm}^{(t)} T}$, where $\hbar \omega_{nm}^{(t)} \equiv E_n^{(t)} - E_m^{(t)}$.

If $\hbar \omega_{nm}^{(t)} >> \left\langle u_m^{(t)} \right| \hat{H}_1 \left| u_n^{(t)} \right\rangle \frac{1}{T}$, i.e. the energy level spacing is large compare to the coupling matrix elements and the rate of change of the Hamiltonian, 1/T, is *slow(adiabatic) enough,* we see that $\dfrac{dc_m^{(t)}}{dt} \approx 0$. Thus, under these conditions we find

$$\Rightarrow \left| \psi(T) \right\rangle = \sum_n c_n^{(0)} e^{-i\phi_n(T)} \left| u_n^{(T)} \right\rangle$$

Adiabatic theorem: If the rate of change of the Hamiltonian is adiabatic (in the sense above), then the *probability* of occupying the n^{th} energy level of $\hat{H}(t)$ is *independent of time*.

(c) Consider the opposite limit. Suppose the Hamiltonian changes “suddenly” (i.e. the inequality is in (b) is reversed). **Show that** (up to an overall phase):

$$\Rightarrow \left| \psi(T) \right\rangle \approx \sum_n c_n^{(0)} \left| u_n^{(0)} \right\rangle = \left| \psi(0) \right\rangle$$

Sudden approximation: If the Hamiltonian is changed *suddenly* (in the sense above), then the state vector does not evolve during this time.

These approximations are extremely useful as tools to “coherently control” the state of the system.

Problem 3: Adiabatic and diabatic (sudden) evolution of a spin (25 points)

An electron with spin 1/2 sits in a magnetic field initially pointing in the $+z$ direction, $\mathbf{B}(t=0) = B_0\,\mathbf{e}_z$. At $t=0$ the electron is in its ground state $|\psi(0)\rangle = |\text{ground state}\rangle$.

The *direction* of the field is then rotated for a time T with angular frequency ω about the $-y$-axis, so that at time $t = T = \pi/(2\omega)$, the magnetic field points in the $+x$-direction. For $t > T$, the field is static:

$$\mathbf{B}(t) = \begin{cases} B_0\,\mathbf{e}_n(t) & 0 \le t \le T \\ B_0\mathbf{e}_x & t \ge T \end{cases}, \text{ with } \mathbf{e}_n(t) = \cos\omega t\,\mathbf{e}_z + \sin\omega t\,\mathbf{e}_x$$

(a) Give an approximate expression for the state vector for times $t > T$ in the two cases:

(i) The magnetic field is rotated "suddenly".

(ii) The magnetic field is rotated "adiabatically".

Please define your notation for any basis set you use to express the state.

What is the condition on ω such that approximations (i) and (ii) hold?

(b) What is the probability, as a function of time, to measure the electron its original state $|\psi(0)\rangle$ for $t > T$, for the two cases (i) and (ii)?

Please explain your results (a picture or two might help).

(c) Find the Heisenberg equations of motion for the vector of spin operators $\hat{\vec{\mathbf{S}}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$, during the time the field is rotated (i.e. for $0 \le t \le T$). Show that the approximate solution in cases (i) and (ii) give the operator $\hat{S}_x$ at time T as:

(i) $\hat{S}_x(T) = \hat{S}_x(0)$ (ii) $\hat{S}_x(T) = \hat{S}_z(0)$

(*Hint:* Go to a noninertial frame, rotating with the magnetic field).

Give a *classical* explanation of these solutions, given the equations of motion.

(d) Use the solutions from part (c) to find the mean value $\hat{S}_x(T)$ or the two cases (i) and (ii). Do these agree with your solution to part (a)?