Physics 521 Fall 2014

Problem Set #6 Due: Tuesday Oct. 14, 2014

Problem 2: The Wigner Function --Quantum Mechanics in Phase Space (20 Points)

(a) We know:

- $\hat{p}$ is the generator of translations in position: $\hat{T}\left(x_0\right) \equiv e^{-i\hat{p}x_0/\hbar}$, $\hat{T}^{\dagger}(x_0)\hat{x}\hat{T}(x_0) = \hat{x} + x_0$
- $-\hat{x}$ is the generator of translations in momentum: $\hat{M}\left(p_0\right) \equiv e^{+i\hat{x}p_0/\hbar}$, $\hat{M}^{\dagger}\left(p_0\right)\hat{p}\,\hat{M}(p_0) \equiv \hat{p} + p_0$

Show that $\hat{D}\left(x_0, p_0\right) \equiv e^{+i(\hat{x}p_0 - \hat{p}x_0)/\hbar}$ is a displacement in *phase space*, i.e., show

$$\hat{D}^{\dagger}\left(x_0, p_0\right)\hat{x}\,\hat{D}\left(x_0, p_0\right) = \hat{x} + x_0\,, \quad \hat{D}^{\dagger}\left(x_0, p_0\right)\hat{p}\,\hat{D}\left(x_0, p_0\right) = \hat{p} + p_0$$

(b) In classical physics we can define a *joint probability density on phase space* $P(x,p)$, normalized so that $\int_{-\infty}^{\infty}\int_{=\infty}^{\infty} dx\,dp\,P(x,p) = 1$. The "marginal" probability density over position or momentum alone are then $P(x) = \int_{=\infty}^{\infty} dp\,P(x,p)$ and $P(p) = \int_{=\infty}^{\infty} dx\,P(x,p)$.

In quantum physics given the state $|\psi\rangle$, $P(x) = \left|\langle x|\psi\rangle\right|^2 = \left|\psi(x)\right|^2$ and $P(p) = \left|\langle p|\psi\rangle\right|^2 = \left|\tilde{\psi}(p)\right|^2$ are the marginal probability densities in x and p respectively. However, we cannot generally write a *joint* probability distribution over phase space because the uncertainty principle forbids it. However we can define a function know as a "quasiprobability" distribution, $W(x,p)$ that has the correct marginals.

Let $W(x,p) \equiv \dfrac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \psi^*(x + x'/2)\psi(x - x'/2)e^{ipx'/\hbar}dx'$ (this is known as the Wigner function)

Show that $\int_{=\infty}^{\infty} dp\,W(x,p) = \left|\psi(x)\right|^2 = P(x)$, $\int_{=\infty}^{\infty} dx\,W(x,p) = \left|\tilde{\psi}(p)\right|^2 = P(p)$

Problem 2: Spreading of a wavepacket in the Schrödinger picture (25 Points)

Consider a normalized Gaussian wavepacket $\psi_{x_0,p_0}(x) = \dfrac{1}{\left(2\pi\sigma^2\right)^{1/4}} e^{-\frac{(x-x_0)^2}{4\sigma^2}} e^{ip_0 x}$.

(a) Sketch $\left|\psi_{x_0,p_0}(x)\right|^2$. Find $\langle\hat{x}\rangle, \langle\hat{p}\rangle, \Delta x, \Delta p$ using the position representation. Is this a minimum uncertainty state?

(b) Find the momentum space wave function $\tilde{\psi}_{x_0,p_0}(p)$. Sketch $\left|\tilde{\psi}_{x_0,p_0}(p)\right|^2$. Check your answer to (a) by calculating $\langle\hat{x}\rangle,\langle\hat{p}\rangle,\Delta x,\Delta p$ using the momentum representation.

(c) Show that $\left|\psi_{x_0,p_0}\right\rangle = \hat{D}(x_0,p_0)\left|\psi_{0,0}\right\rangle$, i.e. $\psi_{x_0,p_0}(x) = \left\langle x\right|\hat{D}(x_0,p_0)\left|\psi_{0,0}\right\rangle$.

(d) Suppose this wave packet describes the initial state of a free particle of a mass *m,* moving in 1D. Find the wave function at a later time according to the form derived in class

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}}\,\tilde{\psi}_{x_0,p_0}(p)\, e^{i(px-\frac{p^2}{2m}t)/\hbar}$$

(e) Use $\psi(x,t)$ to show that $\Delta p^2(t) = \Delta p^2(0)$, $\Delta x^2(t) = \Delta x^2(0) + \dfrac{\Delta p^2(0)}{m^2}t^2$. Show that this is exactly what you would expect *classically* for particle with an initial uncertainty in its position and momentum. Does the wavepacket remain a minimum uncertainty state?

Problem 3: A "Schrödinger-Cat" State (25 points)

Consider a superposition of two Gaussian wavepackets $\psi(x) = \dfrac{N}{\left(2\pi\sigma^2\right)^{1/4}}\left(\mathrm{e}^{-\frac{(x-x_0)^2}{4\sigma^2}} + \mathrm{e}^{-\frac{(x+x_0)^2}{4\sigma^2}}\right)$

P.S. When the separation between the packets is "macroscopic" this is often called a "Schrödinger-cat" state.

(a) Find the normalization constant *N* and show that $N \to \dfrac{1}{\sqrt{2}}$ when $x_0 >> \sigma$.

Sketch $\left|\psi(x)\right|^2$ for $\sigma = 1, x_0 = 4$.

(b) Find the momentum-space wave function $\tilde{\psi}(p)$. Sketch $\left|\tilde{\psi}(p)\right|^2$ for the same parameters. Comment on your sketch.

(c) Find the uncertainties Δx and Δp. Is this a minimum uncertainty state?

(d) Find the Wigner function $W(x,p)$. Make a surface plot of $W(x,p)$ as a function on *x* and *p*. Comment.