Physics 521, Quantum Mechanics I

Problem Set #8:

Cohen-Tannoudji et al. vol. I, Chap. V,

Sakurai: Chap. 2.3, Merzbacher: Chap. 5, 10.6-10.7

Due: Tueday Oct. 28, 2014

Problem 1: Cold atoms trapped in an "optical lattice" (20 Points)

Background:

Neutral atoms are polarizable particles. That is, when interacting with an electromagnetic field, an electric dipole is induced which then interacts with the field. The interaction energy can be written

$$V(\mathbf{x}) = -\frac{1}{2}\mathbf{d}\cdot\mathbf{E}(\mathbf{x}) = -\frac{1}{2}a\,I(\mathbf{x}),$$

where **d** is the induced dipole amplitude, **E** is the electric field, I is the intensity and a is a constant proportional to the polarizability depending on the internal structure of the atom. Assuming a laser field tuned near (but not too close) to some atomic resonance:

$$a = -\frac{\hbar\Gamma^2}{4\Delta}\frac{1}{I_s},$$

where Γ is the "natural linewidth" of the resonance, I_s is the "saturation intensity" characteristic of the transition (i.e. the intensity require for to drive population into the excited state), and $\Delta = \omega_L - \omega_{eg}$ is the detuning of the laser frequency ω_L from the resonance frequency between ground and excited states: $\omega_{eg} = (E_e - E_g)/\hbar$.

Consider now a standing wave whose intensity varies only along one direction (say the z-direction): $I(z) = I_0\sin^2(k_L z)$, where $k_L = 2\pi/\lambda_L$ is the laser vacuum wavelength

$$V(z) = V_0\sin^2(k_L z), \quad V_0 = \frac{\hbar\Gamma^2}{8\Delta}\frac{I_0}{I_s}.$$

Ultra cold atoms can be trapped this periodic potential known as an "optical lattice". We will consider here Cs atoms driven on the so called "D2 resonance", with resonant wavelength $\lambda_{eg} = 852\,\mathrm{nm}$, natural linewidth $\Gamma/2\pi = 5\,\mathrm{MHz}$, and $I_s = 1\,\mathrm{mW/cm^2}$.

(a) Given a laser with $I_0 = I_s$, detuned off resonance so that $\Delta = 5\Gamma$, how cold (in Kelvin) must the atoms be to be trapped?

(b) For ultracold atoms trapped near the minima, the quantum nature of the motion become important. Show that for an atom trapped near the origin, the Hamiltonian has the form.

$$\hat{H} = \frac{\hat{p}^2}{2M} + \frac{1}{2} M\omega_{osc}^2 \hat{z}^2,$$

where M is the mass of ^{133}Cs, and $\hbar\omega_{osc} = 2\sqrt{V_0 E_R}$, where $E_R \equiv (\hbar k_L)^2 / 2M$ is the so-called "recoil energy" (the amount of kinetic energy with which the atom would recoil, if it absorbed a photon with momentum $\hbar k_L$). The atoms vibrational energy levels are then the familiar SHO.

(c) How cold must the atoms be before quantum effects become important? What is the mechanical oscillation frequency of the atoms and what to part of the spectrum does this correspond (i.e. is this a microwave range frequency, etc.)?

(d) If atom scatters a photon (i.e. absorbs and reëmits the photon), it can change it's vibrational energy level. The probability of making such a transition is proportional to the square matrix element:

$$P_{n\to n'} \sim \left|\langle n'|e^{i\delta k\,\hat{z}}|n\rangle\right|^2,$$ where $\delta k = k_f - k_i$ is the difference in the photon wave numbers between the absorbed and emitted waves (along z).

Show that for $\delta k \Delta z_0 \equiv \eta << 1$, where Δz is the ground-state rms width, we have

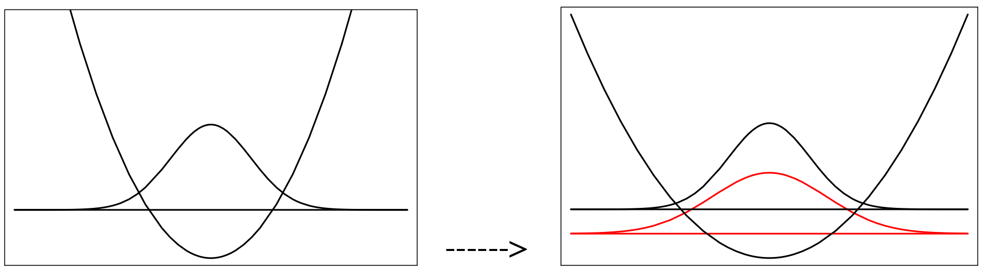
$$P_{0\to n} \sim \delta_{n,0} + \eta^2 \delta_{n,1}$$

This is known as the *Lamb-Dicke* effect: For a well localized atom in the ground state, $\eta << 1$, the probability of scattering a photon that changes the energy level is small.

(e) If $\delta k = k_L$, given the Cs atom initially in the ground state and for the parameters given here, what is the relative probability of the atom scattering a photon and going to the first excited vibrational level?

Problems 2 - Breathing Wavepackets and Squeezed States (25 Points)

Consider again the optical lattice in Problem 1with atoms trapped in the vibrational ground state. Suppose at time t=0, the intensity of the trapping field is *suddenly* decreased so that the atoms now find themselves in a potential with decreased curvature.



t=0$_-$ t=0$_+$

The wavepacket is no longer in the ground state (in the figure above, the red wave packet is the ground state of the new potential with a wider width and lower zero-point energy). As it is no longer a stationary state, it will evolve nontrivially according to the new Hamiltonian. One finds that Gaussian wavepacket "breathes", with its rms widths in position and momentum-space oscillating at frequency 2ω (where ω is the classical oscillation frequency of the new well) in accord with the uncertainty principle.

(a) Without solving the problem completely, at what rate must the intensity be ramped down in order to satisfy the "sudden condition"? Why is the period oscillation of the wave function in the new potential 2ω?

Operator method. Let $\{|n'\rangle\}$ be the set of stationary states associated with the <u>original potential</u> (oscillation frequecy ω'), and $\{|n\rangle\}$ the stationary states of the <u>new potential</u> (oscillation frequecy ω). The ground states are related by a *unitary* transformation:

$$|0'\rangle = \hat{S}(r)|0\rangle, \quad \text{where } \hat{S}(r) = \exp\left\{r(\hat{a}^2 - \hat{a}^{\dagger 2})/2\right\},$$

with $\hat{a}^\dagger$ and $\hat{a}$ the creation and annihilation operators of the new potential.

The state $|0'\rangle$ having a width different from the ground state is known as a *squeezed state*, and for this case a "squeezed ground-state" or "squeezed vacuum"

(b) Show $\hat{S}(r)^\dagger \hat{a}\hat{S}(r) = \cosh(r)\hat{a} - \sinh(r)\hat{a}^\dagger$, $\hat{S}(r)^\dagger \hat{X}\hat{S}(r) = \hat{X}e^{-r}$, $\hat{S}(r)^\dagger \hat{P}\hat{S}(r) = \hat{P}e^{+r}$ where $\hat{X}$ and $\hat{P}$ are the dimensionless position and momentum w.r.t. the *new* potential

and thus show $\langle 0'|\hat{X}|0'\rangle = \langle 0'|\hat{P}|0'\rangle = 0$, $\langle 0'|(\Delta\hat{X})^2|0'\rangle = \frac{e^{-2r}}{2}$, $\langle 0'|(\Delta\hat{P})^2|0'\rangle = \frac{e^{2r}}{2}$.

Sketch the phase space picture of the squeezed vacuum (as we did for the coherent state) with the associated mean, and "quantum uncertainty".

(c) Show that $\hat{S}(r)|X\rangle = \frac{1}{\sqrt{e^{-r}}}|Xe^{-r}\rangle$, where $|X\rangle$ are the eigenstates of $\hat{X}$.

Thus given,

$$\psi(x,0) = \langle x|0'\rangle = u_0'(x) = \frac{1}{(2\pi x_{0'}^2)^{1/4}}\exp(-x^2/4x_{0'}^2)$$

(original ground state, dimensions included),

show that the relation $|0'\rangle = \hat{S}(r)|0\rangle$ makes sense, if we define the "squeezing parameter" r via

$$e^{-2r} = \frac{x_{0'}^2}{x_0^2} = \frac{\omega}{\omega'}$$, with x_0^2, $x_{0'}^2$ the new and old ground state variances respectively

(d) The state for times t>0 is given by, $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$.

Show that $|\psi(t)\rangle = \hat{S}(\zeta(t))|0\rangle$ where $\hat{S}(\zeta(t)) = \exp\left\{(\zeta(t)^*\hat{a}^2 - \zeta(t)\hat{a}^{\dagger 2})/2\right\}$, $\zeta(t) = re^{-2i\omega t}$.

(e) Show that the position and momentum uncertainties vary with time as

$$\left\langle(\Delta\hat{X})^2\right\rangle_t = \frac{e^{-2r}}{2}\cos^2\omega t + \frac{e^{2r}}{2}\sin^2\omega t \qquad \left\langle(\Delta\hat{P})^2\right\rangle_t = \frac{e^{2r}}{2}\cos^2\omega t + \frac{e^{-2r}}{2}\sin^2\omega t$$

Use the phase space picture to explain these results.

(f) ***Extra Credit.*** (5 Points) Show that the initial state has the number state decomposition:

$|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$, $c_n = \frac{1}{\sqrt{n!\mu}}\left(\frac{\nu}{2\mu}\right)^{n/2}\mathrm{H}_n(0)$, where $\mathrm{H}_n(0)$ are the Hermite polynmials evaluated at the origin and and $\mu = \cosh r$ and $\nu = \sinh r$. Find $\langle x|\psi(t)\rangle$ as a Gaussian.

(Hint: You can either use operator methods or the properties of Hermite polynomials)