Physics 521, Quantum Mechanics I
Problem Set #9:
Due: Tuesday Nov. 4, 2014

Problem 1: The (Quantum) Thermal State of the Simple Harmonic Oscillator (25 points)

Consider a particle of mass m in a harmonic potential well of frequency ω in thermal equilibrium. The quantum mechanical state is the mixed state

$$\hat{\rho} = \frac{1}{Z} e^{-\hat{H}/k_B T} \text{, where } \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 = \hbar\omega\left(\hat{a}^\dagger \hat{a} + 1/2\right), \quad Z = tr\left(e^{-\hat{H}/k_B T}\right) .$$

(a) Show that $Z = \dfrac{e^{-\hbar\omega/2k_B T}}{1 - e^{-\hbar\omega/k_B T}}$, and thus $\hat{\rho} = \left(1 - e^{-\hbar\omega/k_B T}\right) \sum_{n=0}^{\infty} e^{-n\hbar\omega/k_B T} \left|n\right\rangle\left\langle n\right|$. Note, since in thermal equilibrium $\hat{\rho}$ is diagonal in the energy basis, all variables are *stationary.*

(b) Show that $\langle \hat{n} \rangle = \dfrac{1}{e^{\hbar\omega/k_B T} - 1}$, the Planck distribution function, and thus show $\langle n | \hat{\rho} | n \rangle = \dfrac{\langle \hat{n} \rangle^n}{\left(1 + \langle \hat{n} \rangle\right)^{n+1}}$, the Bose-Einstein distribution.

(c) Show $\Delta n^2 = \langle \hat{n} \rangle + \langle \hat{n} \rangle^2$. How does this compare to a coherent state?

(d) Show that the probability distribution in dimensionless position and momentum space:

$$\langle X | \hat{\rho} | X \rangle = \frac{1}{\sqrt{2\pi\left(\langle \hat{n} \rangle + 1/2\right)}} \mathrm{e}^{-\frac{X^2}{2(\langle \hat{n} \rangle + 1/2)}}, \quad \langle P | \hat{\rho} | P \rangle = \frac{1}{\sqrt{2\pi\left(\langle \hat{n} \rangle + 1/2\right)}} \mathrm{e}^{-\frac{P^2}{2(\langle \hat{n} \rangle + 1/2)}} \text{ (Gaussians),}$$

and thus $\Delta X^2 = \Delta P^2 = \langle \hat{n} \rangle + 1/2$

Hint: Use $\displaystyle\sum_{n=0}^{\infty} a^n \left(U_n(X)\right)^2 = \frac{1}{\sqrt{\pi(1-a^2)}} \exp\left\{ -\left(\frac{1-a}{1+a}\right) X^2 \right\}$ (Extra credit if you derive this)

(e) In the limit $k_B T << \hbar\omega$ $\hat{\rho} \to |0\rangle\langle 0|$ (the ground state) and in the limit $k_B T >> \hbar\omega$, $\hat{\rho}$ becomes a classical Boltzmann distribution. Show that $\langle x | \hat{\rho} | x \rangle$ and $\langle p | \hat{\rho} | p \rangle$ (with dimensions restored) go the expected position and momentum distributions in these two cases – the classical case is the Maxwell-Bolzmann distribution. Show furthermore that Δx^2 and Δp^2 go to the expected limits, with the classical case following from the equipartition theorem.

Problem 2: Coherent States in the Schrödinger Picture (25 points)

Suppose at time t=0 the state of the system (in the Schrödinger picture) is a coherent state. $|\psi(0)\rangle = |\alpha_0\rangle = \hat{D}(\alpha_0)|0\rangle$, where $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is the phase space displacement operator, and $\alpha_0 = (X_0 + iP_0)/\sqrt{2}$ is the dimensionless complex amplitude.

(a) Show that the position and momentum space wave functions at time t=0 are (up to a negligible constant phase factor):

$$\psi(x,0) = e^{ip_0 x}\, u_0(x - x_0), \qquad \tilde{\psi}(p,0) = e^{-ix_0 p}\, \tilde{u}_0(p - p_0)$$

where $u_0(x)$ and $\tilde{u}_0(p)$ and the ground state wave functions in position and momentum space respectively.

(b) Use these wave functions to show that $\langle\hat{x}\rangle_{t=0} = x_0$, $\langle\hat{p}\rangle_{t=0} = p_0$.

(c) What are $\psi(x,t)$ and $\tilde{\psi}(p,t)$. Please comment.

(d) Find the Wigner function for the state at time *t*, and show that it gives the correct marginals $|\psi(x,t)|^2$ and $|\tilde{\psi}(p,t)|^2$. Plot the function and comment on the nature of the solution.