Physics 521, Quantum Mechanics I
Problem Set #11 - Central Potentials
Read Cohen-Tannoudji Vol. I, Chap VII
Due Friday Dec. 5

Problem 1: 2D Isotropic SHO encore (15 points)

Consider again 2D isotropic SHO descriped by the Hamiltonian

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{1}{2} m\omega^2 (\hat{x}^2 + \hat{y}^2)$$

(a) Show that $\hat{H} = \hbar\omega\left(\hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y + 1\right) = \hbar\omega\left(\hat{N}+1\right)$, where $\hat{a}_x = \frac{\hat{X} + i\hat{P}_x}{\sqrt{2}}$, $\hat{a}_y = \frac{\hat{Y} + i\hat{P}_y}{\sqrt{2}}$ (define dimensionless operators $\left(\hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y\right)$, and the total number operator $\hat{N} = \hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y$.

(b) Show that the energy eigenvectors are

$$\left|n_x\right\rangle \otimes \left|n_y\right\rangle = \frac{\left(\hat{a}_x^\dagger\right)^{n_x} \left(\hat{a}_y^\dagger\right)^{n_y}}{\sqrt{n_x! n_y!}} \left|0_x\right\rangle \otimes \left|0_y\right\rangle$$

with energy eigenvalues are $E_n = \hbar\omega\left(n+1\right)$, where $n = n_x + n_y$, and degeneracy $g_n = n+1$.

(b) Define creation operators $\hat{a}_\pm^\dagger \equiv (\hat{a}_x^\dagger \pm i\hat{a}_y^\dagger)/\sqrt{2}$. Show that the z-component of orbital angular momentum of the particle in the well is $\hat{L}_z = \hbar\left(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-\right)$. Interpret the physical meaning of $\hat{a}_\pm^\dagger \equiv (\hat{a}_x^\dagger \pm i\hat{a}_y^\dagger)/\sqrt{2}$.

(c) Show that $\left|n_+\right\rangle \otimes \left|n_-\right\rangle = \frac{\left(\hat{a}_+^\dagger\right)^{n_+} \left(\hat{a}_-^\dagger\right)^{n_-}}{\sqrt{n_+! n_-!}} \left|0_+\right\rangle \otimes \left|0_-\right\rangle$ is an energy eigenvector, with eigenvalue $E_n = \hbar\omega\left(n+1\right)$, where $n = n_+ + n_-$, and eigenvector of $\hat{L}_z$ with eigenvalue $m\hbar = (n_+ - n_-)\hbar$.

(d) Thus define joint eigenstates of mutually commuting operators $(\hat{N}, \hat{L}_z)$, $\left|n,m\right\rangle$. Show that $\left|n,m\right\rangle = \left|n_+ = \frac{n+m}{2}\right\rangle \otimes \left|n_- = \frac{n-m}{2}\right\rangle$, and from this determine the possible values *m*, and thus the degeneracy of the state.

Problem 2: Coupled Oscillators (15 Points)

Consider two coupled oscillators A and B described by annihilation/creation operators $\hat{a}, \hat{a}^\dagger$ and $\hat{b}, \hat{b}^\dagger$ respectively, satisfying the usual commutation relations. The Hamiltonian describing their dynamics is given to be:

$$\hat{H} = \hbar\omega_A\left(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + 1\right) + \hbar\kappa\left(\hat{a}\hat{b}^\dagger + \hat{b}\hat{a}^\dagger\right)$$ where κ is the "coupling constant".

(a) Show that the solution to the Heisenberg equations of motion are

$$\hat{a}(t) = e^{-i\omega t}\left(\hat{a}(0)\cos\kappa t - i\hat{b}(0)\sin\kappa t\right), \quad \hat{b}(t) = e^{-i\omega t}\left(\hat{b}(0)\cos\kappa t - i\hat{a}(0)\sin\kappa t\right)$$

(b) Suppose that the state of the system (in the Schrödinger picture) at *t=0* is a product state consisting of oscillator A in the first excited state and B in the ground state, $|\psi(0)\rangle = |1\rangle_A \otimes |0\rangle_B$.

Find the state of the composite system at later times, and show that the *reduced* density operator for oscillator A alone in the basis $\{|1\rangle_A, |0\rangle_A\}$ is:

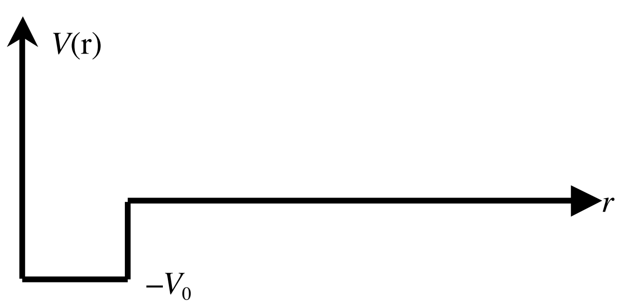
$$\hat{\rho}_A = \begin{bmatrix} \cos^2\kappa t & 0 \\ 0 & \sin^2\kappa t \end{bmatrix}.$$

Please comment on the physical interpretation of your result.

(c) How does the "purity" of the state $\hat{\rho}_A$ vary with time? Comments?

Problem 3: The Finite Spherical Well (15 points)

Consider a spherically symmetric potential, $V(r) = \begin{cases} -V_0 & 0 < r < a \\ 0 & r > a \end{cases}$. Along the radial coordinate, due to the boundary condition at r=0, this is just the half-finite well we studied in Problem Sets 5 and 6.



(a) For $E<0$, the solutions to the T.I.S.E. are bound states. Let $E=-E_b$. Making the ansatz for the stationary state wave functions $\psi_{E,l,m}(r,\theta,\phi)=R_{E,l}(r)Y_l^m(\theta,\phi)$, show that the radial function must have the form,

$$R_{E,l}(r)=\begin{cases} A\ j_l(k_1 r) & 0<r<a \\ A\,\dfrac{j_l(k_1 a)}{h_l^{(1)}(\kappa a)}\,h_l^{(1)}(i\kappa r) & r>a \end{cases}, \quad \text{where } k_1=\sqrt{\frac{2m}{\hbar^2}(V_0-E_b)}\,,\ \kappa=\sqrt{\frac{2m}{\hbar^2}E_b}\ .$$

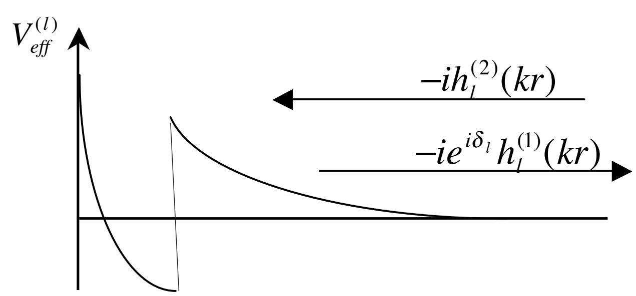
How would you determine A?

(b) Show that the binding energies are determined by the transcendental equation

$$\left(\frac{\frac{d}{dr}(rj_l(k_1 r))}{rj_l(k_1 r)}\right)_{r=a}=\left(\frac{\frac{d}{dr}\left(rh_l^{(1)}(i\kappa r)\right)}{rh_l^{(1)}(i\kappa r)}\right)_{r=a}.$$

Does this reduce to the expected solution of s-states (i.e. $l=0$).

(c) Now consider the unbound states. We seek the scattering phase shift for the asymptotic incoming and outgoing partial waves, as discussed in Lecture.



Show that the phase satisfies the equation

$$\left(\frac{rj_l(qr)}{\frac{d}{dr}[rj_l(qr)]}\right)_{r=a}=\left(\frac{r\left(\cos(\delta_l/2)j_l(kr)-\sin(\delta_l/2)n_l(kr)\right)}{\frac{d}{dr}\left[r\left(\cos(\delta_l/2)j\ (ka)-\sin(\delta_l/2)n_l(kr)\right)\right]}\right)_{r=a},$$

$$\text{where } k=\sqrt{\frac{2m}{\hbar^2}E} \text{ and } q=\sqrt{\frac{2m}{\hbar^2}(E+V_0)}\,.$$

Check that this limits to the expected result for s-wave (l=0).