Physics 522: Graduate Quantum Mechanics II

Lecture 1: Review: Structure of Q.M.

- Basic structure of Quantum theory

Statistical theory - can assign probability to the outcome of experiments. Intrinsic randomness - the deepest question in quantum mechanics

Mathematical Structure: Hilbert space $\mathcal{H}$: Complex vector space with possibly infinite dimensions

-States: Our "state of knowledge" about the system (like a probability assignment

Mathematically:

(Pure) state: Maximal knowledge about a quantum system $\Rightarrow$ "ray" in Hilbert state

$e^{i\phi}|\psi\rangle \equiv$ equivalent class of vectors with irrelevant overall phase

Normalization $\quad ||\psi||^2 = \langle\psi|\psi\rangle = 1$

Observables: What we measure

Mathematically: Hermitian (self-adjoint) operator on Hilbert space $\hat{A} = \hat{A}^\dagger$

Associated with observables are characteristic (eigen) values and vectors:

$$\hat{A}\underset{\substack{\uparrow \\ \text{eigenvector}}}{|a\rangle} = \underset{\nwarrow\,\text{eigenvalue}}{a}\,|a\rangle$$

- Eigenvalues are real
- Nondegenerate eigenvectors are orthogonal $\langle a|a'\rangle = \delta_{aa'}$
- The set of eigenvectors form a basis for $\mathcal{H}$

$\Rightarrow$ resolution of the identity $\sum_a{}' |a\rangle\langle a| = \hat{\mathbb{1}}$

$\Rightarrow$ Any state can be decomposed as a superposition of basis vectors

$$|\psi\rangle = \sum_a{}' |a\rangle\langle a|\psi\rangle = \sum_a{}' c_a\,|a\rangle$$

$c_a = \langle a|\psi\rangle$ = "prabability amplitude"

Note: Relative phase of different c_a's important

$\Rightarrow$ INTERFERENCE of Alternatives

Measurement Prediction

"Born Rule": Probability $p_a = |\langle a|\psi\rangle|^2 = |c_a|^2$

Note if $|\psi\rangle \Rightarrow e^{i\phi}|\psi\rangle$ $\quad p_a \to p_a$

$\Rightarrow$ Overall phase not physical

Given some other observable $\hat{B} = \sum_b b\,|b\rangle\langle b|$

$$\mathcal{P}_b = |\langle b|\psi\rangle|^2 = \left|\sum_a c_a \langle b|a\rangle\right|^2$$

$$= \sum_a \underset{\boxed{P_a}}{\underset{\Downarrow}{|c_a|^2}}\; \underset{\boxed{P_{b|a}\left(\text{conditional probability}\right)}}{\underset{\Downarrow}{|\langle b|a\rangle|^2}} + \sum_{a,a'\neq a} c_a c_{a'}^* \underbrace{\langle a'|b\rangle\langle b|a\rangle}_{\boxed{\text{Interference!}}}$$

- Quantum probabilities do not follow the usual rules of composition
- Quantum amplitudes interfere
- Feynman's heuristic rule:

  In principle indistinguishable alternatives interfere

  Distinguishability $\Rightarrow$ orthogonality

Matrix representation

For a given observable, the dimension of the Hilbert space d, depends on the # of possible alternatives.

Eg. Spin-1/2 particle $d=2$ $\underbrace{\{|\uparrow\rangle, |\downarrow\rangle\}}_{\text{basis}}$

d-dimensional Hilbert space $\Rightarrow$ $\hat{A} = d\times d$ matrix

Matrix elements: $A_{ij} = \langle e_i|\hat{A}|e_j\rangle$

$$\hat{A} \doteq \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1d} \\ \vdots & & & \\ A_{d1} & & \cdots & A_{dd} \end{bmatrix}$$

$|\psi\rangle \Rightarrow$ d-dimen column vector $C_i = \langle e_i|\psi\rangle$

$$|\psi\rangle \Rightarrow \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_d \end{bmatrix}$$

Notation: Direct sum $\equiv$ Subspace

$$\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_B \qquad \mathcal{H}_A = \text{span}\,\{|e_1\rangle \cdot\cdot |e_n\rangle\}$$

$$\mathcal{H}_B = \text{span}\,\{|e_{n+1}\rangle \cdot\cdot |e_d\rangle\}$$

$$\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_d \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} \oplus \begin{bmatrix} C_{n+1} \\ \vdots \\ C_d \end{bmatrix}$$

$\mathcal{H}_A$ and $\mathcal{H}_B$ are subspaces of $\mathcal{H}$

$$\text{Dim}(\mathcal{H}) = \text{Dim}(\mathcal{H}_A) + \text{Dim}(\mathcal{H}_B)$$

$$d = d_A + d_B$$

Direct (Tensor) Product $\equiv$ "subsystem"

E.g. two spin 1/2 particles

$\mathcal{H}_{AB} = h_A \otimes h_B$: Basis $\{|\uparrow\rangle\otimes|\uparrow\rangle, |\uparrow\rangle\otimes|\downarrow\rangle, |\downarrow\rangle\otimes|\uparrow\rangle, |\downarrow\rangle\otimes|\downarrow$

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}_A \otimes \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}_B = \begin{array}{cc} \begin{bmatrix} C_{1A}\,C_{1A} \\ C_{1A}\,C_{2A} \\ C_{2A}\,C_{1B} \\ C_{2A}\,C_{2B} \end{bmatrix} & \begin{array}{c} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{array} \end{array}$$

$$\text{Dim } \mathcal{H}_{AB} = (\dim h_A) \times (\dim h_B)$$

$$d_{AB} = d_A d_B$$

Note: Dimension of Hilbert space grows exponentially with # of sybsystems ⇒ difficulty of many-body physics

Statistical Analysis

- Average (expectation) value

$$\langle \hat{A} \rangle = \sum_a a\, p_a = \sum_a a\, |\langle a|\psi\rangle|^2$$

$$= \langle \psi | \left(\sum_a a |a\rangle\langle a| \right) |\psi\rangle = \langle \psi | \hat{A} | \psi \rangle$$

- Uncertainty (variance about mean)

$$\Delta A^2 = \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$$

- Compatible observables

Heisenberg uncertainty principle

$$\Delta A \, \Delta B \geq \frac{1}{2} \langle [\hat{A}, \hat{B}] \rangle$$

Two observable are "compatible" iff they commute

⇒ Can simultaneous predict outcomes of $\hat{A}$ and $\hat{B}$

~~$\mathrm{Dim}\, \mathcal{H}_{AB} = (\mathrm{Dim}\, h_B) \times (\mathrm{Dim}\, h_B)$~~

~~$d_{AB} = d_A d_B$~~

~~Note: Dimension of Hilbert space grows exponentially with # of spins ⇒ difficulty of many-body physics~~

Physical Connection

Space-time and Hilbert space?

$$\Downarrow$$

$$\boxed{\text{Symmetries}}$$

Noether's theorem: To every geometrical symmetry is associated a conserved physical quantity

Wigner: Symmetries in quantum mechanics are unitary (or anti-unitary as we will see later) operator

Continuous symmetry: Parameterized by a real number λ

$\hat{U}(\lambda)$ with $\hat{U}(0) = \hat{\mathbb{1}}$ (identity)

Near unity infinitesimal $\hat{U}(\epsilon) = \hat{\mathbb{1}} - i\epsilon \hat{G}$

$\hat{G}$ = Hermitian operator = "Generator" of symmetry

Noether theorem: $\nRightarrow$ $\hat{G}$ is conserved if the system is unchanged under symmetry $\hat{U}$

Away from unity: $\hat{U} = e^{-i\lambda\hat{G}}$

<u>Symmetry</u>		<u>Generator</u>
- Spatial Translation:,	$\hat{U}(\vec{x}_0) = e^{-\frac{i}{\hbar}\hat{\vec{p}}\cdot\vec{x}_0}$,	$\hat{\vec{p}}$ (momentum)
- Momentum Translation,	$\hat{U}(\vec{p}_0) = e^{+\frac{i}{\hbar}\hat{\vec{x}}\cdot\vec{p}_0}$,	$-\vec{x}$ (position)
- Time Translation,	$\hat{U}(t_0) = e^{-\frac{i}{\hbar}\hat{H}t_0}$,	$\hat{H}$ $\left(\begin{matrix}\text{energy}\\\text{Hamiltonian}\end{matrix}\right)$
- Rotation about axis,	$\hat{U}(\theta_0, \vec{e}_n) = e^{-\frac{i}{\hbar}(\vec{e}_n\vec{J})\theta_0}$,	$\vec{J}$ $\left(\begin{matrix}\text{angular}\\\text{momentum}\end{matrix}\right)$

<u>Note</u>: Planck's Constant sets the <u>scale</u> of physical symmetries

Distinguishable states separated by "action" $\hbar$

<u>Wave mechanics</u>

Position operator has a continuous spectrum

$$\hat{\vec{x}}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$$

Not normalizable (unphysical) $\langle\vec{x}|\vec{x}'\rangle = \delta^{(3)}(\vec{x}-\vec{x}'$

Resolution $\int d^3\vec{x}\,|\vec{x}\rangle\langle\vec{x}| = \hat{\mathbb{1}}$

State:

$$|\psi\rangle = \int d^3\vec{x}\,|\vec{x}\rangle \underbrace{\langle\vec{x}|\psi\rangle}_{\psi(\vec{x})}$$

("wave function")

$$\langle\psi|\psi\rangle = \int d^3\vec{x}\; |\psi(\vec{x})|^2 = 1$$

(square normalizable function)

Momentum space:

$$\hat{\vec{p}}\,|\vec{p}\rangle = \vec{p}\,|\vec{p}\rangle$$

$$\int d^3\vec{p}\;|\vec{p}\rangle\langle\vec{p}| = \mathbb{1} \qquad \langle\vec{p}|\vec{p}\,'\rangle = \delta^{(3)}(\vec{p}-\vec{p}\,')$$

$$|\psi\rangle = \int d^3\vec{p}\;|\vec{p}\rangle \underbrace{\langle\vec{p}|\psi\rangle}_{\tilde{\psi}(\vec{p})}$$

(momentum-space wave function)

Change of basis: $\langle\vec{x}|\vec{p}\rangle = \frac{1}{(2\pi)^{3/2}}\, e^{i\vec{p}\cdot\vec{x}/\hbar}$

(plane wave)

$$\tilde{\psi}(\vec{p}) = \langle\vec{p}|\psi\rangle = \int d^3\vec{x}\;\langle\vec{p}|\vec{x}\rangle\langle\vec{x}|\Psi\rangle$$

$$= \int \frac{d^3\vec{x}}{(2\pi)^{3/2}}\; e^{-i\vec{p}\cdot\vec{x}/\hbar}\;\psi(\vec{x})$$

(Fourier transform)

Canonical commutators: $[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$

$$\Rightarrow \boxed{\Delta x\,\Delta p_x \geq \frac{\hbar}{2} \qquad \text{etc.}}$$