Physics 522: Quantum Mechanics II

Problem Set #8 Solutions

Problem 1: Driven SHO

$$\hat{H} = \hat{H}_0 + \hat{H}_{int}(t)$$

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 = \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)$$

$$\hat{H}_{int}(t) = -F(t)\,\hat{x} \qquad \left[F(t) = -\frac{\partial \hat{H}}{\partial x} \ \text{(Gradient of potential)} \right]$$

(a) At $t \to -\infty$ $|\psi\rangle = |n=0\rangle$, ground state

Find Probability to be in n^{th} excited state at $t \to +\infty$

First, use dimension variable. Define $x_c = \sqrt{\frac{2\hbar}{m\omega}}$

$$\Rightarrow \hat{x} = x_c \hat{X} \qquad \hat{X} = \frac{\hat{a} + \hat{a}^\dagger}{2}$$

$$\Rightarrow \hat{H}_{int} = f(t)\left(\hat{a} + \hat{a}^\dagger\right) \qquad \text{where} \quad f(t) = -x_c F(t)$$

Interaction picture $\Rightarrow \hat{H}_{int}^{(I)} = f(t)\left(\hat{a}\, e^{-i\omega t} + \hat{a}\, e^{+i\omega t}\right)$

According to first-order perturbation, the probability of transition for $|0\rangle \Rightarrow |n\rangle$ over ∞ time is

$$P_{n\leftarrow 0} = \left| \frac{-i}{\hbar} \int_{-\infty}^{\infty} \langle n | \hat{H}_{int}^{(I)}(t) | 0 \rangle \, dt \right|^2$$

$\Rightarrow$ In first order perturbation theory

$$P_{n\leftarrow 0} = \left| \int_{-\infty}^{\infty} \frac{-i}{\hbar} f(t) \left(\langle n|\hat{a}|0\rangle e^{-i\omega t} + \langle n|\hat{a}^{\dagger}|0\rangle e^{+i\omega t} \right) \right|$$

$$= \left| \int_{-\infty}^{\infty} \frac{-i}{\hbar} f(t)\, e^{i\omega t} \right| \delta_{n,1}$$

$$\boxed{P_{n\leftarrow 0} = |\tilde{g}(\omega)|^2 \delta_{n,1}}$$

where $\tilde{g}(\omega) = \int_{-\infty}^{\infty} \frac{-i}{\hbar} f(t)\, e^{i\omega t}$

(b) Now solve exactly. The propagator in the interaction picture

$$\hat{U}^{(I)} = \underbrace{\mathcal{T}}_{\text{time ordering}} \left[\exp\left[\int_{-\infty}^{\infty} dt \; \frac{-i}{\hbar} f(t) \left(\hat{a}\, e^{-i\omega t} + \hat{a}^{\dagger} e^{i\omega t} \right) \right] \right]$$

Note: $\left[\hat{H}^{(I)}_{int}(t_1), \hat{H}^{(I)}_{int}(t_2) \right] = f(t_1) f(t_2)\, e^{-i\omega(t_1 - t_2)} - f(t_2) f(t_1)\, e^{+i\omega(t_1 - t_2)}$

Since this is a c-number (not an operator), the time-ordering is unimportant (it only changes the overall phase

$$\Rightarrow \hat{U}^{(I)} = e^{i\phi} \exp\left[\tilde{g}(\omega)\, \hat{a}^{\dagger} - \tilde{g}^{*}(\omega)\, \hat{a} \right]$$

Now, using the Baker-Campbell-Hausdorf theorem

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}$$

$$\text{if } [\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$$

$\Rightarrow$ with $\hat{A} = \tilde{g}(\omega)\hat{a}^\dagger \quad \hat{B} = \tilde{g}^*(\omega)\hat{a}$

$$\Rightarrow \hat{U}^{(I)} = e^{i\phi} e^{-\frac{1}{2}|\tilde{g}(\omega)|^2} e^{\tilde{g}(\omega)\hat{a}^\dagger} e^{\tilde{g}^*(\omega)\hat{a}}$$

Transition probability:

$$P_{n\leftarrow 0} = |\langle n| \hat{U}^{(I)} |0\rangle|^2$$

$$= e^{-|\tilde{g}(\omega)|^2} |\langle n| e^{\tilde{g}(\omega)\hat{a}^\dagger} |0\rangle|^2$$

$$= e^{-|\tilde{g}(\omega)|^2} \Big| \sum_m \frac{1}{m!} \underbrace{\langle n| (\tilde{g}(\omega))^m a^{\dagger m} |0\rangle}_{\sqrt{m!}\,(\tilde{g}(\omega))^m \delta_{n,m}} \Big|^2$$

$$\Rightarrow \boxed{P_{n\leftarrow 0} = \frac{1}{n!} |\tilde{g}(\omega)|^{2n} e^{-|\tilde{g}(\omega)|^2}}$$

For $|\tilde{g}(\omega)|^2 \ll 1 \qquad P_{1\leftarrow 0} = |\tilde{g}(\omega)|^2$

in agreement

(c) Using the time evolution operator in the interaction picture:

$$\hat{X}^{(I)}(t) = \hat{U}^{\dagger(I)} \hat{X}(0) \hat{U}^{(I)}$$

$$= \hat{X}(0) + \mathrm{Re}(\tilde{g}(\omega))$$

this follows from:

$$\hat{U}^{(I)} = \exp\{i \,\mathrm{Re}(\tilde{g}(\omega)) \hat{P} + i \,\mathrm{Im}(\tilde{g}(\omega)) \hat{X}]$$

where $\hat{a} = \hat{X} + i\hat{P}$

- $\hat{P}$ is the generator of displacements in $\hat{X}$

(d) Classically, the amplitude of a driven oscillator is

$$x(t) = \int_0^t G(t-t') \frac{F(t')}{m} dt' + x(0)$$

where $G(\tau)$ is the Green's function of the SHO $\left(\frac{d^2}{d\tau^2} + \omega_0^2\right) G(\tau) = \delta(\tau)$

$$\Rightarrow \quad G(\tau) = \frac{1}{\omega} \sin\omega\tau$$

$$\therefore \; x(t) = \frac{-i}{2m\omega}\left(\int_0^t e^{i\omega t'} F(t')\, dt'\right) e^{-i\omega t} + c.c. + x(0)$$

In the rotating frame (taking out the free evolution

$$x^{(I)}(t) = \frac{i}{2m\omega_0} \int_0^t e^{i\omega t'} F(t')\, dt' + c.c. + x(0)$$

Thus,

$$x^{(I)} = \frac{i\hbar}{2m\omega_0}\int_0^t e^{i\omega t'}\,\frac{F(t')}{\hbar}\,dt' + c.c. + x(0)$$

$$= \frac{1}{2}\Big[\underbrace{\int \frac{-i}{\hbar}\,x_c\,F(t')\,e^{i\omega t'}\,dt'}_{\tilde{g}(\omega)} + c.c.\Big] + x(0)$$

$$\Rightarrow \boxed{x^{(I)}(t) = x(0) + \mathrm{Re}\left(\tilde{g}(\omega)\right)}$$ As before

These agree perfectly. This is because everything is linear. According to Ehrenfest's theorem, classical + quantum trajectories are the same if linear

Rabi vs. Ramsey

Problem 3

Two level atom. In RWA (in rotating frame)

$$\hat{H}_{eff} = -\frac{\hbar\Delta}{2}\hat{\sigma}_z - \frac{\hbar\Omega}{2}\hat{\sigma}_x = -\frac{\hbar}{2}\vec{\Omega}\cdot\hat{\vec{\sigma}}$$

$$\vec{\Omega} = \Delta\vec{e}_z + \Omega\vec{e}_x = \tilde{\Omega}\,\vec{e}_n$$

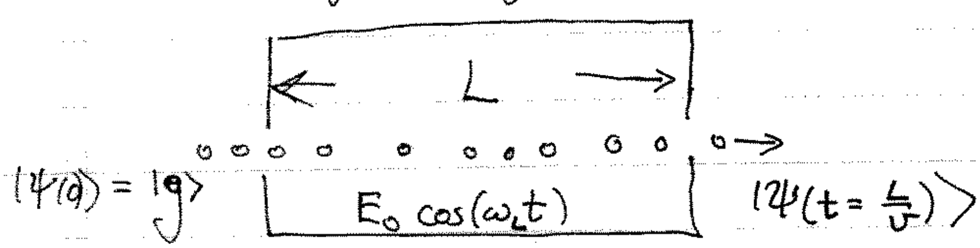
$$\tilde{\Omega} = \sqrt{\Omega^2+\Delta^2} \qquad \vec{e}_n = \cos\theta\,\vec{e}_z + \sin\theta\,\vec{e}_x \qquad \cos\theta = \frac{\Delta}{\tilde{\Omega}}, \quad \sin\theta = \Omega/\tilde{\Omega}$$

Unitary evolution (in rotating frame)

$$\hat{U}(t) = e^{-i\hat{H}_{eff}t/\hbar} = e^{i\frac{\tilde{\Omega}t}{2}\hat{\sigma}_n} = \cos\left(\frac{\tilde{\Omega}t}{2}\right)\hat{\mathbb{1}} + i\sin\left(\frac{\tilde{\Omega}t}{2}\right)\hat{\sigma}_n$$

$$= \cos\left(\frac{\tilde{\Omega}t}{2}\right)\hat{\mathbb{1}} + i\sin\frac{\tilde{\Omega}t}{2}\left(\cos\theta\,\hat{\sigma}_z + \sin\theta\,\hat{\sigma}_x\right)$$

(i) Rabi geometry



$$\Omega = d_{eg}E_0/\hbar$$

(a) Mono-energic beam with velocity v, length chosen such that $\Omega\frac{L}{v} = \pi$ (i.e. π-pulse for atoms on resonance)

Now we have the Rabi solution:

$$|\Psi(t)\rangle = \hat{U}(t)\,|g\rangle =$$

$$= \left(\cos\left(\frac{\tilde{\Omega}t}{2}\right) - i\cos\theta\,\sin\left(\frac{\tilde{\Omega}t}{2}\right)\right)|g\rangle + i\sin\theta\,\sin\frac{\tilde{\Omega}t}{2}\,|e\rangle$$

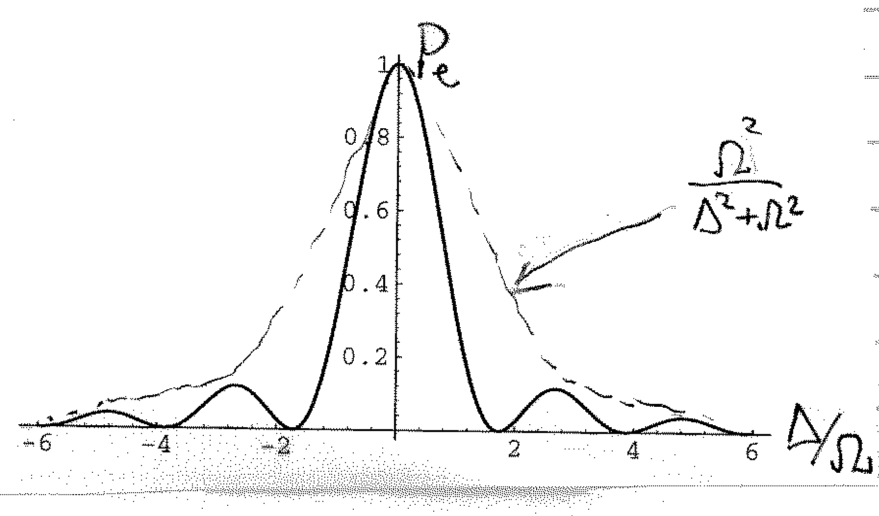
Thus the probability to be in the excited state after a time $t = \frac{L}{v}$ is

$$P_e = \sin^2\theta\,\sin^2\left(\frac{\tilde{\Omega}L}{2v}\right) = \frac{\Omega^2}{\Delta^2+\Omega^2}\,\sin^2\left(\frac{\sqrt{\Omega^2+\Delta^2}\,L}{2v}\right)$$

$$\Rightarrow \boxed{P_e = \frac{1}{1+\left(\frac{\Delta}{\Omega}\right)^2}\,\sin^2\left(\left(1+\frac{\Delta^2}{\Omega^2}\right)^{1/2}\frac{\pi}{4}\right)}$$

(where I used $\Omega L/v = \pi$)

Plot of P_e as a function of $\Delta = \omega_L - \omega_{eg}$ in units of Ω



Linewidth $\Delta\omega \sim \Omega = \frac{\pi}{T}$ where $T = \frac{L}{v}$

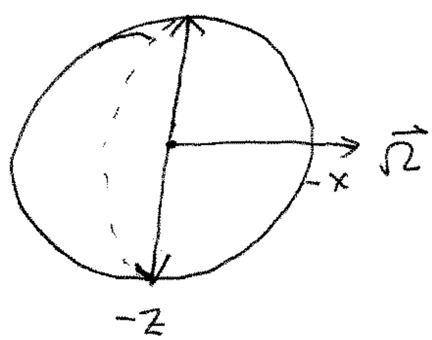
(d) Continued

We see that the linewidth is on the order

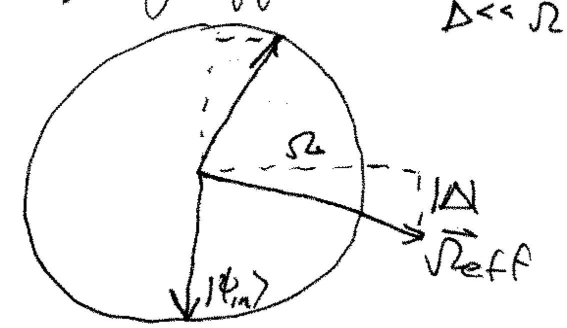
$$\Delta\omega \cong \Omega = \frac{\pi}{T} \quad \text{where} \quad T = \frac{L}{v} = \text{interaction time}$$

On the Bloch-sphere

<u>Resonance $\Delta = 0$</u>



Slightly off-resonance $\Delta \ll \Omega$



$$\vec{\Omega}_{eff} = \Omega \vec{e}_x + \Delta \vec{e}_z$$

From simple geometry we see that $P_e = \left| \frac{\Omega}{\sqrt{\Delta^2 + \Omega^2}} \right|^2$ (for $\tilde{\Omega} T = \pi$)

It is clear from these sketches that when Δ is on the order of Ω population transfer to the excited state decreases substantially. Since we have fixed $\Omega T = \pi$ we get the most sensity to Δ by making T longer and thus Ω smaller

(b) Now suppose the atoms have a distribution of velocities characteristic of thermal beams

$$f(v) = 2\frac{v^3}{v_0^4} e^{-\frac{v^2}{v_0^2}} \quad \text{where} \quad v_0 = \sqrt{\frac{2k_BT}{m}}$$

Note: for this distribution

- Most probable velocity: $v_p = 1.22\, v_0$
- Average velocity: $\bar{v} = 1.33\, v_0$
- rms velocity: $\sqrt{\Delta v^2} = 1.42\, v_2$

We have for a <u>fixed</u> velocity and detuning

$$P_e(\Delta, v) = \frac{1}{1+\left(\frac{\Delta}{\Omega}\right)^2} \sin^2\left(\left(1+\frac{\Delta^2}{\Omega^2}\right)^{1/2}\frac{\Omega L}{2v}\right)$$

At zero detuning (on resonance): $P_e(0,v) = \sin^2\left(\frac{L\Omega}{2v}\right)$

Now we must average over the velocity distribution:

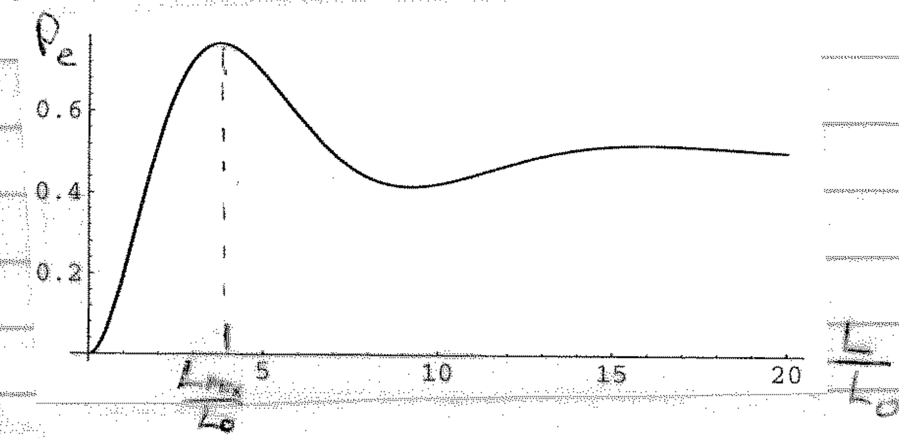
$$P_e = \int_0^\infty dv\, f(v)\, P_e(0,v) = \int_0^\infty dx\, f(x)\, \sin^2\left(\frac{\theta_0}{2x}\right)$$

where $x \equiv \frac{v}{v_0}$ (dimensionless velocity)

and $\theta_0 = \frac{\Omega L}{v_0}$ (Angle turned by the Bloch vector for atoms traveling at v_0 and ω_L on resonance)

<u>Note</u>: <u>Always</u> reëxpress in terms of dimensionless variables

Plot (calculated with Mathematica)



$\left(L_0 = \frac{v_0}{\Omega}\right)$

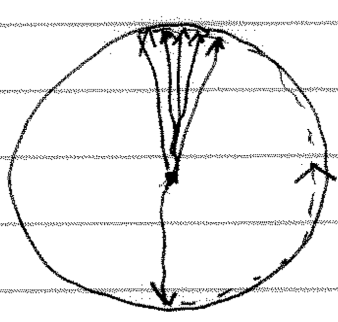
The maximum P_e occurs at $\Theta_0 = \frac{\Omega L_{max}}{v_0} \cong 3.77$

or $\frac{\Omega L^{max}}{v_0} \approx 1.20\pi$. We can understand this

from the fact that the most probable speed is

$v_p \cong 1.2 v_0 \Rightarrow \Omega \frac{L_{max}}{v_p} \approx \pi$

On the Bloch-sphere:



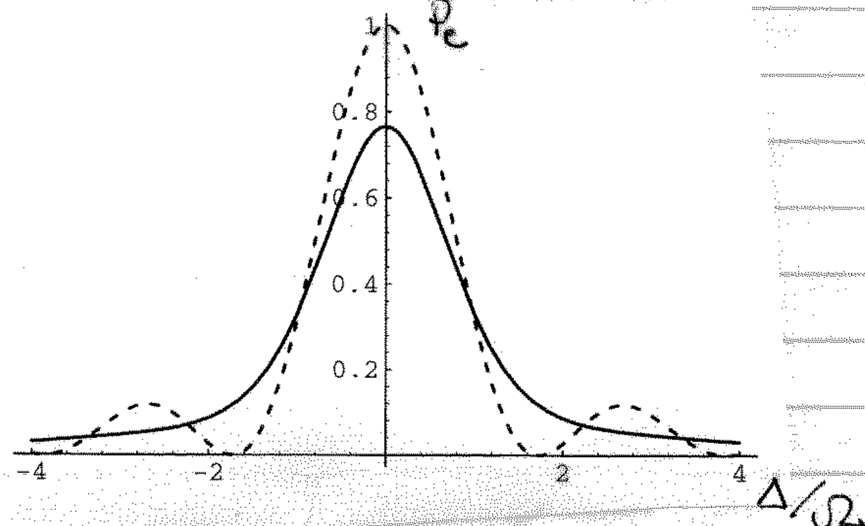
Distribution of velocities
$\Rightarrow$ Distribution of interaction times
$\Rightarrow$ Distribution of rotation angles

This is an example of "inhomogenous" broadening. The "damped" Rabi ossillations plotted above are not due to dissipation. Rather they are due to the spread in rotation angle due to an inhomoginity in the sample (here velocity).

Now set $\theta_0 = 1.2\pi \approx 3.8$, and $\Delta \neq 0$

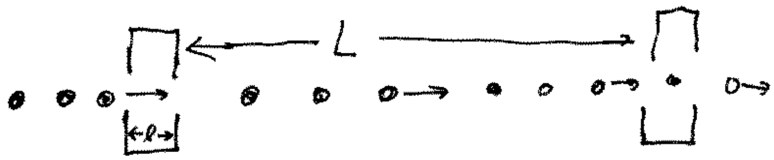
$$\Rightarrow P_e = \frac{1}{1+\left(\frac{\Delta}{\Omega}\right)^2} \int_0^\infty \sin^2\left(\left(1+\frac{\Delta^2}{\Omega^2}\right)^{1/2} \frac{\theta_0}{2} \frac{1}{x}\right) \left(2x^3 e^{-x^2}\right) dx$$

Plot as a function of Δ/Ω with $L = L_{max}$
(Solid curve)



For reference I have added the curve for mono-energetic atom (dashed curve). We see that the spread in velocities leads to a broadening of the resonance lineshape.

(ii) Ramsey separated zone method



(c) Given $|\psi(0)\rangle = |g\rangle$ and $|\Delta| \ll \Omega$

In the interaction regions: $\tilde{\Omega} \cong \Omega \qquad \Theta \approx \pi/2$

$$\hat{U}_{int}(t) \cong \cos\left(\frac{\Omega t}{2}\right)\hat{\mathbb{1}} + i\sin\left(\frac{\Omega t}{2}\right)\hat{\sigma}_x = e^{i\frac{\Omega t}{2}\hat{\sigma}_x}$$

In the free zone: $\tilde{\Omega} = \Delta \qquad \Theta = 0$

$$\hat{U}_{free}(t) = \cos\left(\frac{\Delta t}{2}\right)\hat{\mathbb{1}} + i\sin\frac{\Delta t}{2}\,\hat{\sigma}_z = e^{i\Delta t/2\,\hat{\sigma}_z}$$

(free precession about $-\vec{e}_z$ axis with frequency Δ (in the rotating frame))

$$\therefore\ |\psi(\tau = \tfrac{\ell}{v})\rangle = \cos\left(\frac{\Omega\tau}{2}\right)|g\rangle + i\sin\left(\frac{\Omega\tau}{2}\right)|e\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|g\rangle + i|e\rangle\right)$$

$T = \frac{L}{v}$

$$|\psi(\tau + T)\rangle = \hat{U}_{free}(T)\,|\psi(\tau)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\Delta T/2}|g\rangle + i e^{i\Delta T/2}|e\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\frac{\Delta L}{2v}}|g\rangle + i e^{i\frac{\Delta L}{2v}}|e\rangle\right)$$

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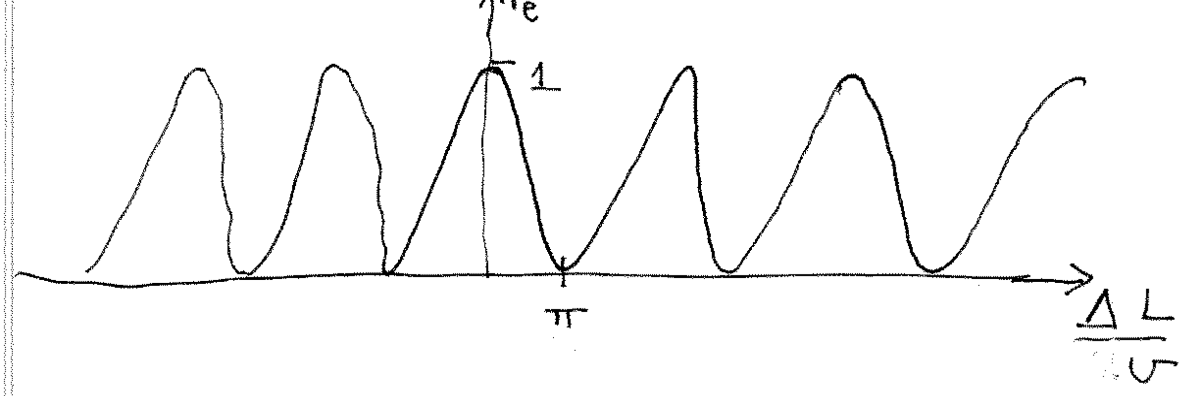
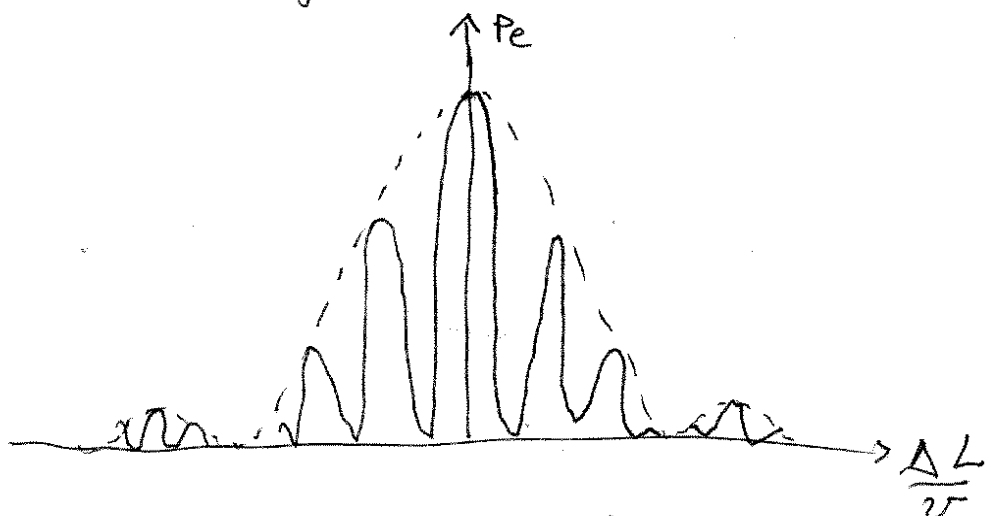
(d) $P_e(t_{Final}) = \cos^2\left(\frac{\Delta L}{2v}\right)$



This is known as "Ramsey fringes"

<u>Note</u>: We have assumed $\Delta \ll \Omega$. For the general solution we would have



The envelope of the fringe pattern is the Rabi pattern of part (a) due to the transit time brodening of width $\Delta\omega \sim \frac{v}{l}$

The fringes themselves have a width on the order $\Delta\omega \sim \frac{v}{L}$ which can be much smaller

$$|\psi(2\tau+T)\rangle = \hat{U}_{int}(\tau)\,|\psi(\tau+T)\rangle$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\frac{\Delta L}{2v}}\,\hat{U}_{int}(\tau)|g\rangle + i e^{i\frac{\Delta L}{2v}}\,\hat{U}_{int}(\tau)|e\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\frac{\Delta L}{2v}}\frac{1}{\sqrt{2}}\left(|g\rangle + i|e\rangle\right) + i\frac{e^{i\frac{\Delta L}{2v}}}{\sqrt{2}}\left(|e\rangle + i|g\rangle\right)\right)$$

$$= \frac{1}{2}\left(e^{-i\frac{\Delta L}{2v}} - e^{i\frac{\Delta L}{2v}}\right)|g\rangle + \frac{i}{2}\left(e^{i\frac{\Delta L}{2v}} + e^{-i\frac{\Delta L}{2v}}\right)|e\rangle$$

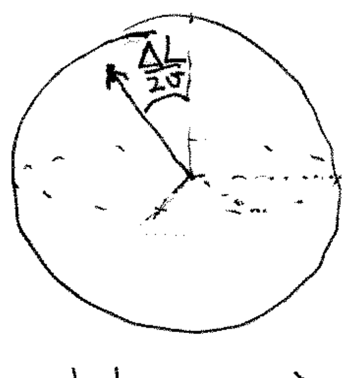
$$\Rightarrow \boxed{|\psi(2\tau+\tau)\rangle = -i\sin\left(\frac{\Delta L}{2v}\right)|g\rangle + i\cos\left(\frac{\Delta L}{2v}\right)|e\rangle}$$

Block sphere in Rotating frame

$|\psi(0)\rangle$ $\qquad$ $|\psi(\tau)\rangle$ $\qquad$ $|\psi(\tau+T)\rangle$

angle of precession $\Delta \cdot T$

$\Longrightarrow$ Final Zone

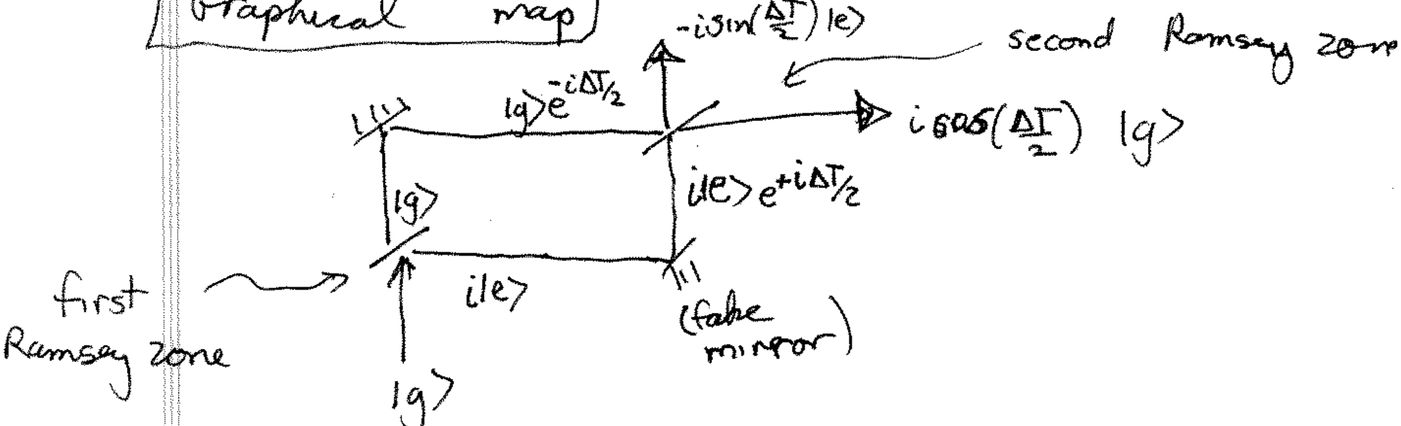


$|\psi(2\tau+T)\rangle$

Final vector rotated towards north-pole. Final angle $\theta = \pi - \frac{\Delta L}{2v}$

(e) It is clear from the plots of part (d) the the Ramsey geometry results in a kind of interference pattern. We can think of the two "branches" of the wavefunction, $|e\rangle$ or $|g\rangle$, as the two "arms" of an interferometer. The Ramsey zones produce $\pi/2$ pulse are like "beam splitters". In the free interaction zone the two branches of the interferometer pick up a relative phase of ΔT.

Graphical map



The key to the Ramsey ~~meth~~ meathod is to make T as big as possible. Then we can measure extremely small detunings Δ because a measurable phase will accumulate (shown in the third mapping on the Bloch sphere). This is the basic physics behind the atomic clock. The idea is to lock a "local oscillator" to to the natural oscillator of a two level atom (Two hyperfine levels of cesium). When the frequency of the oscillator is on resonance we're there!