Physics 522

Quantum Mechanics II

Problem Set #9 - Solutions

Problem 5: Photon absorption cross-section / scattering

Given a rate of absorption R_{abs}, we have the absorbed power $P_{abs} = \hbar\omega R_{abs}$

The absorption cross-section is thus

$$\sigma_{abs} = \frac{P_{abs}}{I_{inc}} \leftarrow \text{incident intensity}$$

Given an incident plane wave, according to Fermi's Golden, the absorption rate (transition $|g\rangle \to |e\rangle$)

$$R_{abs} = \frac{2\pi}{\hbar^2} |\langle e| \hat{H}^{(+)}_{int} |g\rangle|^2 g(\omega)$$

Normalized atomic line shape at ω

$$\hat{H}^{(+)}_{int} = -\frac{1}{2}\hat{\vec{d}}\cdot\vec{E}_0 = -\frac{1}{2}(\hat{\vec{d}}\cdot\vec{\epsilon}_L)\, E_0$$

polarization, amplitude

$$\Rightarrow \sigma_{abs}(\omega) = \frac{\hbar\omega}{I} R_{abs} = \frac{\hbar\omega}{I_{inc}} \left| \frac{\langle e|\hat{\vec{d}}\cdot\vec{\epsilon}_L|g\rangle}{2} \right|^2 E_0^2\, g(\omega)$$

$$= \frac{4\pi^2}{\hbar c} |\langle e|\hat{\vec{d}}\cdot\vec{\epsilon}_L|g\rangle|^2\, \omega\, g(\omega)$$

having used $I_{inc} = \frac{c}{4\pi}\langle|\vec{E}(\vec{x},t)|^2\rangle = \frac{c}{8\pi}E_0^2$

(using natural line shape) : $\sigma_{abs}(\omega) = \frac{2\pi}{\hbar c} |\langle e|\hat{\vec{d}}\cdot\vec{\epsilon}_L|g\rangle|^2 \frac{\omega\Gamma}{\Delta^2 + \frac{\Gamma^2}{4}}$

where the natural line shape $g(\omega) = \dfrac{\Gamma/2\pi}{(\omega-\omega_{eg})^2 + \frac{\Gamma^2}{4}}$

with $\Gamma = \dfrac{4}{3\hbar}\left(\dfrac{\omega}{c}\right)^3 |\langle e|\vec{d}|g\rangle|^2$ the natural linewidth $= (\text{lifetime})^{-1}$

$$\Rightarrow \sigma_{abs}(\omega) = \frac{8\pi\omega}{\hbar c} \frac{|\langle e|\hat{\vec{d}}\cdot\vec{\epsilon}_L|g\rangle|^2}{\Gamma} \frac{1}{\left(1+\frac{4\Delta^2}{\Gamma^2}\right)}$$

$$\boxed{\sigma_{abs}(\omega) = \frac{\sigma_0}{\left(1+\frac{4\Delta^2}{\Gamma^2}\right)}}$$

$\left(\sigma_0 = \dfrac{8\pi\omega}{\hbar c}\dfrac{|\langle e|\hat{\vec{d}}\cdot\vec{\epsilon}_L|g\rangle|^2}{\Gamma} = 8\pi\alpha\dfrac{\omega}{\Gamma}|\langle e|\hat{\vec{x}}\cdot\vec{\epsilon}_L|g\rangle|^2 \right.$ the resonant cross-section $\Big)$

(b) We now consider two possibility: polarized vs. unpolarized.

(i) Assume the atom is prepared in a well defined m-state, and the light field is polarized with $\vec{\epsilon}_L = \vec{e}_q$

$$\Rightarrow \sigma_0 = \frac{8\pi\omega}{\hbar c}\frac{|\langle e, J_e\|d\|g, J_g\rangle|^2}{\Gamma}|\langle J_e\ m_g+q|1\ q\ J_g\ m_g\rangle|^2$$

having used the W.E.T.

$$\langle e; J_e M_e|\, d_q\, |g; J_g M_g\rangle$$

$$= \langle e, J_e\|d\|g, J_g\rangle\langle J_e M_e|1 q\ J_g M_g\rangle$$

$M_e = M_g + q$

Aside: As discussing in P.S. #3 (problem 3), for multilevel atoms we must sum over all final states to ~~obta~~ obtain the natural linewidth

$$\Gamma = \frac{4}{3\hbar} k^3 \sum_{M_g} |\langle e; J_e M_e | \vec{d} | g; J_g M_g \rangle|^2$$

M_e arbitrary

$$\Gamma = \frac{4}{3\hbar} k^3 |\langle e; J_e \| d \| g; J_g \rangle|^2$$

$\Rightarrow$ For a polarized atom

$$\sigma_0 = \frac{8\pi\omega}{\hbar c} \left(\frac{3\hbar c^3}{4\omega^3} \right) |\langle J_e\ m_g+q | 1\ q\ J_g\ M_g \rangle|^2$$

$$\boxed{\sigma_0 = 6\pi \lambda\!\!\!^{-2} |\langle J_e, m_g+q | 1\ q\ J_g\ m_g \rangle|^2}$$

where $\lambda\!\!\!^{-} = \frac{\lambda}{2\pi} = \frac{1}{k} = \frac{c}{\omega}$

Note: $(\sigma_0)_{max} = 6\pi \lambda\!\!\!^{-2}$ for transition with unit C-G coeff.

(ii) Unpolarized case: Average over all polarizations

$$|\langle e | \vec{d} \cdot \vec{\epsilon}_\lambda | g \rangle|^2 \Rightarrow \frac{1}{3} \sum_\lambda |\langle e | \vec{d} \cdot \vec{\epsilon}_\lambda | g \rangle|^2$$

$$= \frac{1}{3} |\langle e | \vec{d} | g \rangle|^2$$

$$\Rightarrow \sigma_0 = (6\pi \lambda\!\!\!^{-2}) \frac{1}{3} = 2\pi \lambda\!\!\!^{-2}$$

Problem 2: Two-photon transitions

We want to drive a hydrogen atom from $1s \to 100s$. This requires a two-photon transition since this is $l=0 \to l=0$, which is completely forbidden. To achieve this transition we use two photons so that

$$\hbar\omega_1 + \hbar\omega_2 \approx E_{100s} - E_{1s}$$

(a) From class, we saw that from second-order time-dependent perturbation theory, if $\omega_1 = \omega_2 = \omega_L$

$$P_{f\leftarrow i}(t) = \left| \sum_m \frac{\langle f|\hat{H}^{(+)}_{int}|m\rangle\langle m|\hat{H}^{(+)}_{int}|i\rangle}{(E_m - E_i) - \hbar\omega_L} \; \frac{2e^{i\delta t/2}\sin(\delta t/2)}{\delta} \right|^2$$

$$\text{where } \delta = 2\omega_L - \left(\frac{E_f - E_i}{\hbar}\right)$$

We interpret this as the transition from $i \to f$ in two steps, $i \to m \to f$, where the intermediate step is known as a "virtual transition". The weight of each term $|m\rangle\langle m|$ depends on a resonant denominator. Thus, the terms that dominate are those for which the photon is near resonance. In a two-photon transition, if one of the photons is close to resonance with an intermediate level, it is this level that is the only important contribution to the sum.

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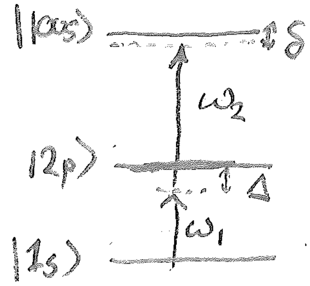
Here, one laser is relatively close to the $1s \to 2p$ transition, and thus the $2p$ state is the only state that contributes in any substantial way in the virtual transition



Here $\delta \equiv \omega_1 + \omega_2 - \left(\frac{E_{100s} - E_{1s}}{\hbar}\right)$

$$\Delta \equiv \omega_1 - \left(\frac{E_{2p} - E_{1s}}{\hbar}\right)$$

Keeping only the near-resonant terms:

$$P_{100s \leftarrow 1s}(t) = \left| \frac{\langle 100s | \hat{H}_2^{(+)} | 2p \rangle \langle 2p | \hat{H}_1^{(+)} | 1s \rangle}{E_{2p} - E_{1s} - \hbar\omega_1} \right|^2 \frac{4\sin^2(\delta t/2)}{\delta^2}$$

Here $\hat{H}^{(+)}_{\substack{1\\2}} = -\hat{\vec{d}} \cdot \vec{E}_{\substack{1\\2}} \frac{e^{-i\omega_{\substack{1\\2}} t}}{2}$

$$\Rightarrow \boxed{P_{100s \leftarrow 1s}(t) = \left(\frac{\Omega_1 \Omega_2}{2\Delta\delta}\right)^2 \sin^2\left(\frac{\delta t}{2}\right)}$$

where $\hbar\Omega_1 = \langle 2p | \hat{\vec{d}} | 1s \rangle \cdot \vec{E}$, $\hbar\Omega_2 = \langle 100s | \hat{\vec{d}} | 2p \rangle \cdot \vec{E}_2$

(b) Restricting to the near resonance terms, (equivalent to the rotating wave approximation) the interaction Hamiltonian is

$$\hat{H}_{int} = -\frac{\hbar\Omega_1}{2}\left(|2p\rangle\langle 1s| e^{-i\omega_1 t} + |1s\rangle\langle 2p| e^{+i\omega_1 t}\right)$$
$$-\frac{\hbar\Omega_2}{2}\left(|100s\rangle\langle 2p| e^{-i\omega_2 t} + |2p\rangle\langle 100s| e^{+i\omega_2 t}\right)$$

Let us adopt the short-hand

$$|a\rangle = |1s\rangle, \quad |b\rangle = |2p\rangle, \quad |c\rangle = |100s\rangle$$

$$\Rightarrow \hat{H}_{int} = -\frac{\hbar\Omega_1}{2}\left(|b\rangle\langle a| e^{-i\omega_1 t} + |a\rangle\langle b| e^{+i\omega_1 t}\right)$$
$$-\frac{\hbar\Omega_2}{2}\left(|c\rangle\langle b| e^{-i\omega_2 t} + |b\rangle\langle c| e^{+i\omega_2 t}\right)$$

Go to the interaction picture:

$$\hat{H}^{(I)}_{int} = \hat{U}_0^{\dagger}\hat{H}_{int}\hat{U}_0, \quad \text{where } \hat{U}_0 = e^{-\frac{i}{\hbar}\hat{H}_A t}$$

$$\Rightarrow \hat{H}^{(I)}_{int} = -\frac{\hbar\Omega_1}{2}\left(|b\rangle\langle a| e^{-i\Delta t} + |a\rangle\langle b| e^{+i\Delta t}\right)$$
$$-\frac{\hbar\Omega_2}{2}\left(|c\rangle\langle b| e^{-i(\delta-\Delta)t} + |b\rangle\langle c| e^{+i(\delta-\Delta)t}\right)$$

Here I used $\Delta = \omega_1 - \left(\frac{E_{2p} - E_{1s}}{\hbar}\right)$

and $\omega_2 - \left(\frac{E_{100s} - E_{2p}}{\hbar}\right) = \omega_2 + \omega_1 - \left(\frac{E_{100s} - E_{1s}}{\hbar}\right)$
$- \omega_1 + \left(\frac{E_{2p} - E_{1s}}{\hbar}\right) = \delta - \Delta$

Given this, we can write the time-dependent Schrödinger Eqn in the interaction picture

$$\frac{d}{dt}|\Psi^{(I)}\rangle = -\frac{i}{\hbar}\hat{H}^{(I)}_{int}|\Psi^{(I)}\rangle$$

Expanding: $|\Psi^{(I)}\rangle = c_a(t)|a\rangle + c_b(t)|b\rangle + c_c(t)|c\rangle$

$$\Rightarrow \frac{d}{dt}c_j(t) = -\frac{i}{\hbar}\langle j|\hat{H}^{(I)}_{int}|\Psi^{(I)}\rangle$$

$$\therefore \frac{d}{dt}c_a(t) = \frac{i\Omega_1}{2}e^{i\Delta t}c_b(t)$$

$$\frac{d}{dt}c_b(t) = \frac{i\Omega_1}{2}e^{-i\Delta t}c_a(t) + \frac{i\Omega_2}{2}e^{i(\delta-\Delta)t}c_c(t)$$

$$\frac{d}{dt}c_c(t) = \frac{i\Omega_2}{2}e^{-i(\delta-\Delta)t}c_b(t)$$

(c) When $|\Delta| \gg \Omega_1, \Omega_2, \delta$ we can "adiabatically eliminate" $c_b(t)$

$$c_b(t) = \frac{i\Omega_1}{2}\int_0^t dt'\, e^{-i\Delta t'}c_a(t') + \frac{i\Omega_2}{2}\int_0^t dt'\, e^{+i(\delta-\Delta)t'}c_c(t')$$

Adiabatic evolution $\Rightarrow c_a(t), c_c(t)$ vary much more slowly than $e^{\pm i\Delta t}$ $\Rightarrow$ these can be factored outside the integral.

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Thus,

$$C_b(t) \approx \frac{-\Omega_1}{2\Delta}\left(e^{-i\Delta t}-1\right) c_a(t) - \frac{\Omega_2}{2(\Delta-\delta)}\left(e^{-i(\Delta-\delta)t}-1\right) c_c(t)$$

$$\Rightarrow \quad \frac{d}{dt} c_a(t) = \frac{-i\Omega_1^2}{4\Delta}\left(1-e^{i\Delta t}\right) c_a(t) - i\,\frac{\Omega_1\Omega_2}{4\Delta}\left(e^{i\delta t}-e^{i\Delta t}\right) c_c(t)$$

$$\frac{d}{dt} c_c(t) = \frac{-i\Omega_2^2}{4(\Delta-\underbrace{\delta}_{\text{neglect}})}\left(1-e^{i(\Delta-\delta)t}\right) c_c(t) - \frac{i\Omega_1\Omega_2}{4\Delta}\left(e^{-i\delta t}-e^{+i(\Delta-\overbrace{\delta}^{\text{neglect}})t}\right) c_a(t)$$

Dropping the terms that oscillate like $e^{\pm i\Delta t}$ (fast)

$$\Rightarrow \quad \boxed{\begin{aligned} \frac{d}{dt} c_a(t) &= -i\,\frac{V_a}{\hbar}\, c_a(t) - i\,\frac{\Omega_{eff}}{2}\, e^{i\delta t}\, c_c(t) \\ \frac{d}{dt} c_c(t) &= -i\,\frac{V_c}{\hbar}\, c_c(t) - \frac{i\Omega_{eff}}{2}\, e^{-i\delta t}\, c_a(t) \end{aligned}}$$

where

$$\boxed{\begin{gathered} V_a = \frac{\hbar\Omega_1^2}{4\Delta} \qquad V_c = \frac{\hbar\Omega_2^2}{4\Delta} \\ \Omega_{eff} = \frac{\Omega_1\Omega_2}{2\Delta} \end{gathered}}$$

We thus must solve the system of O.D.E.'s

$$\frac{d}{dt}\begin{bmatrix} c_a \\ c_c \end{bmatrix} = \frac{-i}{\hbar} \underbrace{\begin{bmatrix} V_a & \frac{\hbar\Omega_{eff}}{2} e^{i\delta t} \\ \frac{\hbar\Omega_{eff}}{2} e^{-i\delta t} & V_c \end{bmatrix}}_{\hat{H}_{eff}(t)} \begin{bmatrix} c_a \\ c_c \end{bmatrix}$$

To solve this, it's easiest to go to the rotating frame by the transformation

$$\hat{U}_0 = e^{+i\delta t\,\hat{\sigma}_z/2} = \begin{bmatrix} e^{+i\delta t/2} & 0 \\ 0 & e^{-i\delta t/2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \tilde{c}_a \\ \tilde{c}_c \end{bmatrix} \equiv \hat{U}_0 \begin{bmatrix} c_a \\ c_c \end{bmatrix} = \begin{bmatrix} c_a e^{i\delta t/2} \\ c_c e^{-i\delta t/2} \end{bmatrix}$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} \tilde{c}_a \\ \tilde{c}_c \end{bmatrix} = \frac{-i}{\hbar}\left(\hat{U}_0^{\dagger} H_{eff}\, \hat{U}_0 + i\hbar \frac{\partial \hat{U}_0}{\partial t} \hat{U}_0^{\dagger}\right) \begin{bmatrix} \tilde{c}_a \\ \tilde{c}_c \end{bmatrix}$$

$$= -\frac{i}{\hbar} \tilde{H}_{eff} \begin{bmatrix} \tilde{c}_a \\ \tilde{c}_b \end{bmatrix}$$

Where

$$\tilde{H}_{eff} = \begin{bmatrix} V_a - \frac{\hbar\delta}{2} & \frac{\hbar\Omega_{eff}}{2} \\ \frac{\hbar\Omega_{eff}}{2} & V_c + \frac{\hbar\delta}{2} \end{bmatrix}$$

$$\tilde{H}_{eff} = \frac{V_a + V_c}{2}\hat{\mathbb{1}} - \hbar\frac{\delta_{eff}}{2}\hat{\sigma}_z + \hbar\frac{\Omega_{eff}}{2}\hat{\sigma}_x$$

Here I have defined

$$\delta_{eff} = \delta - \frac{(V_a + V_c)}{\hbar}$$

And the $\vec{\sigma}$ matrices are for the two-levels with $|\Uparrow\rangle = |a\rangle$, $|\Downarrow\rangle = |c\rangle$

The term proportional to $\hat{1}$ effect the overall phase as is thus negligible. The remaining effective Hamiltonian for the two-level system in the rotating frame is

$$\tilde{H}_{eff} = -\frac{\hbar\delta_{eff}}{2}\hat{\sigma}_z + \frac{\hbar\Omega_{eff}}{2}\hat{\sigma}_x$$

This is isomorphic to the problem of Rabi oscillations studied in class. With the state at the initial time

$$|\tilde{\psi}(0)\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} \tilde{C}_a(t) &= \cos\left(\frac{\tilde{\Omega}t}{2}\right) + i\frac{\delta_{eff}}{\tilde{\Omega}}\sin\left(\frac{\tilde{\Omega}t}{2}\right) \\ \tilde{C}_c(t) &= i\frac{\Omega_{eff}}{\tilde{\Omega}}\sin\left(\frac{\tilde{\Omega}t}{2}\right) \\ &\text{where } \tilde{\Omega} \equiv \sqrt{\Omega_{eff}^2 + \delta_{eff}^2} \end{aligned}}$$

Back in the "lab frame", giving the desired result

$$\boxed{\begin{aligned} C_a(t) &= e^{-i\delta t/2}\,\tilde{C}_a(t) \\ C_c(t) &= e^{+i\delta t/2}\,\tilde{C}_c(t) \end{aligned}}$$

(e) From this solution, we see the follow interpretation of the variables:

- $\Omega_{eff} = \dfrac{\Omega_1 \Omega_2}{2\Delta}$ = two-photon Rabi frequency on resonance

- $\delta_{eff} = \delta + \left(\dfrac{V_c - V_a}{\hbar}\right)$ = effective two-photon detuning

- $V_{ac} = \dfrac{\hbar \Omega_{1,2}^2}{4\Delta}$ = level shift induced on $|a\rangle$ and $|c\rangle$ (ac-Stark shift), which adds to the bare detuning

- $\tilde{\Omega} = \sqrt{\Omega_{eff}^2 + \delta_{eff}^2}$ = two-photon Rabi frequency off resonance

(f) The probability to make the transition $|a\rangle \to |c\rangle$ in an off resonance Rabi oscillation

$$P_{c \leftarrow a}(t) = |\langle c|\Psi(t)\rangle|^2 = \frac{\Omega_{eff}^2}{\tilde{\Omega}^2} \sin^2\left(\frac{\tilde{\Omega} t}{2}\right)$$

to lowest order in Ω_1, Ω_1 $\quad \tilde{\Omega} \approx \delta_{eff} \approx \delta$

$$\Rightarrow \boxed{P_{c \leftarrow a}(t) \approx \left(\frac{\Omega_{eff}}{\delta}\right)^2 \sin^2\left(\frac{\delta t}{2}\right) = \left(\frac{\Omega_1 \Omega_2}{2\Delta\delta}\right)^2 \sin^2\left(\frac{\delta t}{2}\right)}$$

as in part (a)