Physics 522. Quantum Mechanics II

Problem Set #1 – Review of Central Potentials and Hydrogen

Due Tuesday, Feb. 1, 2011

Problem 1: Dipole Matrix Elements in Hydrogen. (10 points)

In the "electric-dipole approximation", the interaction energy between an atomic system and an external electric field is $\hat{H}_{\text{int}} = -\hat{\mathbf{d}} \cdot \mathbf{E}$, where $\hat{\mathbf{d}} = -e\hat{\mathbf{x}}$ is the electric dipole operator, $\hat{\mathbf{x}}$ is the observable corresponding to the electron-proton separation, and $\mathbf{E}$ is the electric field (which we treat classically).

(a) Consider the matrix elements between two stationary states of the hydrogen atom:

$$\langle n', l', m' | \hat{H}_{\text{int}} | n, l, m \rangle = e \langle n', l', m' | \hat{\mathbf{x}} \cdot \mathbf{E} | n, l, m \rangle$$

Show that parity conservation implies that the matrix element vanishes exactly unless $(-1)^{l'-l} = -1$.

Note This is not *strictly* true. Due to the electroweak interaction a very small amount of parity symmetry breaking occurs in atomic systems; the physical consequences of this symmetry breaking have been accurately measured and constitute one of the better tests of the "standard model" of particle-physics.

(b) For electric fields pointing along *each* of the Cartesian directions $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ of the relative coordinate vector **x**, find the matrix elements of between the $1s$ state and the three $2p$ states. Please comment on your results. (*Hint*, use the spherical basis and use the orthogonality of the spherical harmonics).

The above results may then be generalized to yield the familiar electric dipole "selection rules" of spectroscopy $\Delta l = \pm 1$, $\Delta m = 0, \pm 1$. In general, the relative magnitudes of the matrix elements are given by *Clebsch-Gordan coefficients*.
This is an example of the Wigner-Eckart theorem...stay tuned.

Problem 2: Hydrogenic atoms and atomic units. (10 points)

Consider the "hydrogenic" atoms - that is bound-states of two oppositely charged particles:

(i) The hydrogen atom: Binding of an electron and proton.

(ii) Heavy ion: Single electron bound to a nucleus of mass M, charge Ze (say Z=50).

(iii) Muonium: Muon bound to a proton

(iv) Positronium: Bound state of an electron and a positron (anti-electron)

(a) For each, using the charges, *reduced* mass, and the unit $\hbar$, determine the characteristic scales of:

Length, energy, time, momentum, and internal electric field. Please give numerical values as well as the expressions in terms of the fundamental constants.

(b) Now add the speed of light c into the mix. Find characteristic velocity in units of c, magnetic field, and magnetic moment. Show that for the particular case of hydrogen the characteristic velocity is $v/c = \alpha = \frac{e^2}{\hbar c}(cgs) \approx \frac{1}{137}$, the "fine-structure" constant, and that the Bohr radius, Compton wavelength, and "classical electron radius", differ by powers of α according to,

$$r_{class} = \alpha \frac{\lambda_{compton}}{2\pi} = \alpha^2 a_0$$

Problem 3: The Infinite Spherical Well and Partial waves (10 Points)

The purpose of this problem is to review some aspects of the radial equation in 3D wave mechanics and lay some ground work for our study of scattering later in the semester. Most of the problem is worked out in textbooks, but it's worthwhile for you to go through the details yourself.

(a) Consider a mass *m* particle in a spherically symmetric potential well of radius R_0, constant inside, and with infinite walls:

$$V(r) = \begin{cases} 0 & r < R_0 \\ \infty & r > R_0 \end{cases}$$

Show that stationary state wave functions are bound states:

$$\psi_{n,l,m}(r,\theta..\phi) = A_{nlm}\; j_l(k_n r) Y_l^m(\theta,\phi),$$

where $j_l(x)$ is a spherical Bessel function, $j_l(k_n R_0) = 0$, and $(A_{nlm})^2 = 2 / R_0^3 \left[j_l'(k_n R_0) \right]^2$.
What are the energy eigenvalues

(b) In the limit $R_0 \to \infty$ the particle is *free,* with *k* becoming a continuous parameter, the energy spectrum $E(k) = (\hbar k)^2 / 2m$, wave functions $\psi_{k,l,m}(r,\theta..\phi) = A(k)\; j_l(kr) Y_l^m(\theta,\phi)$.
These solutions are known as "partial waves".
But we had said free particle eigenstates were plane waves? Why is this also a set of free particle eigenfunctions (Hint: Consider complete set of commuting operators).

(c) Consider now the "inside out" version of (a),

$$V(r) = \begin{cases} \infty & r < R_0 \\ 0 & r > R_0 \end{cases}$$

This constitutes a "hard" sphere off which waves can scatter.
Show that the spectrum now is continuous associated with unbound solutions,

$$\psi_{k,l,m}(r,\theta..\phi) = \left(A_l(k)\; j_l(kr) + B_l(k) n_l(kr) \right) Y_l^m(\theta,\phi),$$

where $B_l(k) / A_l(k) = -j_l(kR_0) / n_l(kR_0)$

(d) For "s-waves", i.e. *l*=0, show that the reduced radial wave function in (c) is the same as the free space wave function, but phase shifted,

$$u_{hard-sphere}(kr,\theta,\phi) = u_{free}(kr + \delta_0)$$

where $\delta_0 = -kR_0$. Explain graphically why this makes sense.