Physics 522. Quantum Mechanics II

Problem Set #6 – Tensors and the Wigner-Eckart Theorem

Due Thursday, Mar. 31, 2011

Problem 1: Landé Projection Theorem (10 Points)

The Landé Projection Theorem (LPT) is a special case of the Wigner-Eckart Theorem for the case that the initial and final state are the same angular momentum. It states:

$$\langle \alpha'; jm' | \hat{\mathbf{V}} | \alpha; jm \rangle = \frac{\langle \alpha'; j | \hat{\mathbf{J}} \cdot \hat{\mathbf{V}} | \alpha; j \rangle}{j(j+1)} \langle jm' | \hat{\mathbf{J}} | jm \rangle, \text{ where } \hat{\mathbf{V}} \text{ is a vector operator w.r.t. } \hat{\mathbf{J}}.$$

(a) Give a geometric interpretation of this in terms of a vector picture.
(b) To prove this theorem, take the following steps (do not give verbatim, Sakurai's derivation):

- (i) Show that $\langle \alpha'; j,m | \hat{\mathbf{J}} \cdot \hat{\mathbf{V}} | \alpha; j,m \rangle = \langle \alpha'; j \| J \| \alpha'; j \rangle \langle \alpha'; j \| V \| \alpha; j \rangle$, independent of m.
- (ii) Use this to show, $|\langle \alpha; j \| J \| \alpha; j \rangle|^2 = j(j+1)$ *independent of* α.
- (iii) Show that $\langle jm' | 1q\, jm \rangle = \langle jm' | \hat{J}_q | jm \rangle / \sqrt{j(j+1)}$.
- (iv) Put it all together to prove the LPT.

(c) As an application of the LPT, consider the Zeeman interaction for hydrogen.

$$\hat{H}_{\text{int}} = -\hat{\vec{\mu}} \cdot \mathbf{B},$$

where the magnetic dipole operators is $\hat{\vec{\mu}} = -\mu_B (g_l \hat{\mathbf{L}} + g_s \hat{\mathbf{S}})$, with $g_l = 1$, $g_s = 2$.

If the Zeeman effect is small compared to the fine-structure, but ignoring hyperfine structure, use the LPT to show that in a state nL_J, the magnetic moment has the form,

$$\hat{\vec{\mu}} = -g_J \mu_B \hat{\vec{\mathbf{J}}}, \text{ where } g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)} \text{ is known as the Landé g-factor.}$$

$$\textit{Hint:} \text{ Use } \mathbf{J} \cdot \mathbf{L} = \mathbf{L}^2 + \frac{1}{2}\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right) \text{ and } \mathbf{J} \cdot \mathbf{S} = \mathbf{S}^2 + \frac{1}{2}\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right).$$

(d) Estimate the magnetic field strength at which the Zeeman interaction becomes on the order of the fine structure splitting between the $2p_{1/2}$ and $2p_{3/2}$ state in hydrogen.

Problem 2: Natural lifetimes of Hydrogen (10 points)

Though in the absence of any perturbation, an atom in the excited state will stay there forever (it is a stationary state), in reality it will "spontaneously decay" to the ground state. Fundamentally this occurs because the atom is always perturbed by 'vacuum fluctuations" in the electro-magnetic field. We will find later in the semester, the spontaneous emission rate on a dipole allowed transition from initial excited state $|\psi_e\rangle$ to all allowed grounds states $|\psi_g\rangle$ is ,

$$\Gamma = \frac{4}{3\hbar} k^3 \sum_g \left| \langle \psi_g | \hat{\mathbf{d}} | \psi_e \rangle \right|^2, \text{ where } k = \omega_{eg} / c \text{ is the emitted photon's wave number.}$$

Consider now hydrogen including fine-structure. For a given sublevel, the spontaneous emission rate is

$$\Gamma_{(nLJM_J)\to(n'L'J')} = \frac{4}{3\hbar} k^3 \sum_{M_J'} \left| \langle n'L'J'M_J' | \hat{\mathbf{d}} | nLJM_J \rangle \right|^2 .$$

(a) Show that the spontaneous emission rate is *independent* of the initial M_J. Explain this result physically.

(b) Calculate the lifetime ($\tau = 1/\Gamma$) of the $2p_{1/2}$ state in seconds.

Problem 3: Light-shift for multilevel atoms (10 points)

Consider a general monochromatic electric field $\mathbf{E}(\mathbf{x},t) = \mathrm{Re}\left(\mathbf{E}(\mathbf{x})e^{-i\omega_L t}\right)$, driving an atom near resonance on the transition, $\left|g;J_g\right\rangle \to \left|e;J_e\right\rangle$, where the ground and excited manifolds are each described by some total angular momentum J with degeneracy $2J+1$. The generalization of the AC-Stark shift is the light-shift operator acting on the $2J_g+1$ dimensional ground manifold:

$$\hat{V}_{LS}(\mathbf{x}) = -\frac{1}{4}\mathbf{E}^*(\mathbf{x})\cdot\hat{\vec{\alpha}}\cdot\mathbf{E}(\mathbf{x}).$$

Here $\hat{\vec{\alpha}} = -\dfrac{\hat{\mathbf{d}}_{ge}\hat{\mathbf{d}}_{eg}}{\hbar\Delta}$ is the atomic polarizability tensor operator, where $\hat{\mathbf{d}}_{eg} \equiv \hat{P}_e\hat{\mathbf{d}}\hat{P}_g$ is the dipole operator, projected between the ground and excited manifolds; the projector onto the excited manifold is, $\hat{P}_e = \sum_{M_e=-J_e}^{J_e}\left|e;J_e,M_e\right\rangle\left\langle e;J_e,M_e\right|$, and similarly for the ground.

(a) By expanding the dipole operator in the spherical basis, **show** that the polarizability operator can be written,

$$\hat{\vec{\alpha}} = \tilde{\alpha}\left(\sum_{q,M_g}\left|C_{M_g}^{M_g+q}\right|^2\vec{\mathbf{e}}_q\left|g;J_g,M_g\right\rangle\left\langle g;J_g,M_g\right|\vec{\mathbf{e}}_q^* + \sum_{q\neq q',M_g}C_{M_g+q-q'}^{M_g+q}C_{M_g}^{M_g+q}\,\vec{\mathbf{e}}_{q'}\left|g;J_g,M_g+q-q'\right\rangle\left\langle g;J_g,M_g\right|\vec{\mathbf{e}}_q^*\right)$$

$$\text{where } \tilde{\alpha} \equiv -\frac{\left|\left\langle e;J_e\|d\|g;J_g\right\rangle\right|^2}{\hbar\Delta} \text{ and } C_{M_g}^{M_e} \equiv \left\langle J_e\,M_e\middle|1q\,J_g\,M_g\right\rangle.$$

(b) Consider a polarized plane wave, with complex amplitude of the form, $\mathbf{E}(\mathbf{x}) = E_1\vec{\varepsilon}_L e^{i\mathbf{k}\cdot\mathbf{x}}$ where E_1 is the amplitude and $\vec{\varepsilon}_L$ the polarization (possibly complex). For an atom driven on the transition $\left|g;J_g=1\right\rangle \to \left|e;J_e=2\right\rangle$ and the cases (i) linear polarization along z, (ii) positive helicity polarization, (iii) linear polarization along x, **find** the eigenvalues and eigenvectors of the light-shift operator. Express the eigenvalues in units of $V_1 = -\frac{1}{4}\tilde{\alpha}\left|E_1\right|^2$. Please ***comment*** on what you find for cases (i) and (iii). **Repeat** for $\left|g;J_g=1/2\right\rangle \to \left|e;J_e=3/2\right\rangle$ and **comment**.

(c) A deeper insight into the light-shift potential can be seen by expressing the polarizability operator in terms of irreducible tensors. **Show** that we get the sum of scalar, vector, and rank-2 irreducible tensor interaction,

$$\hat{V}_{LS} = -\frac{1}{4}\left(\left|\mathbf{E}(\mathbf{x})\right|^2\hat{\alpha}^{(0)} + (\mathbf{E}^*(\mathbf{x})\times\mathbf{E}(\mathbf{x}))\cdot\hat{\alpha}^{(1)} + \mathbf{E}^*(\mathbf{x})\cdot\hat{\alpha}^{(2)}\cdot\mathbf{E}(\mathbf{x})\right),$$

$$\text{where } \hat{\alpha}^{(0)} = \frac{\hat{\mathbf{d}}_{ge}\cdot\hat{\mathbf{d}}_{eg}}{-3\hbar\Delta},\ \hat{\alpha}^{(1)} = \frac{\hat{\mathbf{d}}_{ge}\times\hat{\mathbf{d}}_{eg}}{-2\hbar\Delta},\ \hat{\alpha}_{ij}^{(2)} = \frac{1}{-\hbar\Delta}\left(\frac{\hat{\mathbf{d}}_{ge}^i\hat{\mathbf{d}}_{ge}^j + \hat{\mathbf{d}}_{ge}^i\hat{\mathbf{d}}_{ge}^j}{2} - \frac{\hat{\mathbf{d}}_{ge}\cdot\hat{\mathbf{d}}_{eg}}{3}\delta_{ij}\right).$$