Physics 522. Quantum Mechanics II

Problem Set #8 – Time Dependent Evolution

Due Thursday, April 28, 2011

Problem 1:. Driven Harmonic Oscillator (10 points)

Consider the problem of a simple harmonic oscillator driven by a classical force $F(t)$. Quantum mechanically, the Hamiltonian takes the form,

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}(t)$$

where $\hat{H}_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$ and $\hat{H}_{\text{int}}(t) = -F(t)\hat{x}$. Note, the force, $F(t) = -\frac{\partial}{\partial x}\hat{H}_{\text{int}}(t)$ as required.

(a) Suppose at $t \to -\infty$ the system is in the ground state. Using first order perturbation theory, find the probability to be in the n^{th} excited state at $t \to +\infty$.

(b) Now solve the problem *exactly* and show that the transition probability is

$$P_{n\leftarrow 0} = \frac{1}{n!}\left|\tilde{F}(\omega)\right|^2 e^{-\left|\tilde{F}(\omega)\right|^2},$$

where $\tilde{F}(\omega) = \int_{-\infty}^{+\infty} F(t) e^{i\omega t}\, dt$ is the Fourier transform of the force at the frequency of the oscillator.

Show that for a weak force, the result reduces to the transition probability you found in part (a).

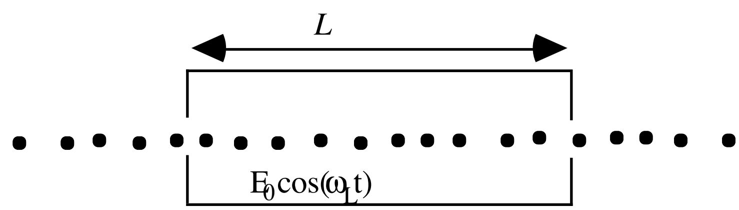
(c) Use the exact time evolution operator to find the mean position of the oscillator in the asymptotic limit.

(d) Now do the same problem *classically,* i.e., determine the position of the oscillator according to classical mechanics under the action of this force. Compare to part (c) and comment on your result.

Problem 2: Spin-Resonance: Rabi vs. Ramsey (20 points)

The technique of measuring transition frequencies with magnetic resonance was pioneered by I. I. Rabi in the late 30's. It was modified by Ramsey (his student) about 10 years later, and now serves as the basis for atomic clocks and the SI definition of the second. All precision atomic measurements, including modern atom-interferometers and quantum logic gates in atomic systems, have at their heart a Ramsey type geometry.

(i) Rabi resonance geometry. Consider a beam of two-level atoms with transition frequency ω_{eg}, passing through an "interaction zone" of length *L*, in which they interact with a monochromatic laser field of frequency ω_L.

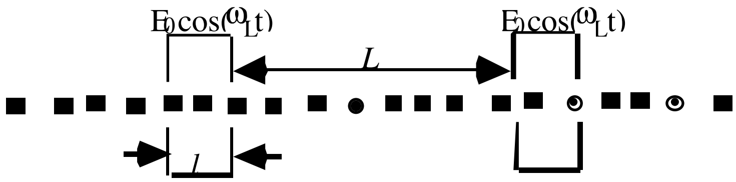


(a) Suppose all the atoms start in the ground-state $|g\rangle$, and have a well defined velocity *v*, chosen such that $\Omega L / v = \pi$, where $\hbar\Omega = d_{eg}E_0$. Plot the probability to be in the excited state $|e\rangle$, P_e, as a function of driving frequency ω_L, neglecting spontaneous emission (what is the condition that we can do this?). What is the linewidth? Explain your plot in terms of the Bloch-sphere.

(b) Now suppose the atoms have a distribution of velocities characteristic of thermal beams: $f(v) = \frac{2}{v_0^4} v^3 \exp(-v^2 / v_0^2)$, where $v_0 = \sqrt{2k_B T / m}$. Plot P_e as a function of *L* for Δ=0, (you may need to do this numerically). At what *L* is it maximized - explain? Also plot as in (a), P_e as a function of ω_L with $L = L_{max}$. What is the linewidth? Explain in terms of the Bloch-sphere.

(ii) Ramsey separated zone method

As you have seen in parts (a)-(b), assuming one can make the velocity spread sufficiently small, the resonance linewidth is limited by the interaction time *L/v*. This is known as "transit-time broadening" and is a statement of the time-energy uncertainty principle. Unfortunately, if we make *L* larger and larger other inhomogenities, such as the amplitude of the driving field come into play. Ramsey's insight was that one can in fact "break up" the π-pulse given to the atoms into two $\pi/2$-pulses in a time $\tau = l/v$ (i.e. $\Omega\tau = \pi / 2$), separated by *no interaction* for a time $T = L/v$. The free interaction time can then made *much* longer.



(b) Given a mono-energetic atoms with velocity v, internal state $|\psi(0)\rangle = |g\rangle$, and field at a detuning $|\Delta| << \Omega$ so that $\tilde{\Omega} \approx \Omega$ find:
$|\psi(\tau = l/v)\rangle$, $|\psi(\tau + T = (l + L)/v)\rangle$, $|\psi(2\tau + T = (2l + L)/v)\rangle$
and show that mapping of the state on the Bloch-sphere.

(c) Plot $P_e(t_{final} = 2\tau + T)$ as a function of ω_L. Plot also for the case of finite spread in velocity as in (b). What is the linewidth?

(d) A Ramsey separated zone geometry is often described as a kind of "interferometer." Explain why this makes sense.