Physics 522. Quantum Mechanics II

Problem Set #10 – Time Dependent Perturbations

Due Thursday, May 5 2011

Problem 1: Photon absorption cross-section and photon scattering (10 points)
Given a laser beam at frequency ω_L incident of an atom, we can define the absorption cross-section in the usual way,

$$\sigma_{abs}(\omega_L) = \frac{P_{abs}}{I_{inc}},$$

where I_{inc} is the incident intensity and P_{abs} is the absorbed power Assume a "weak" incident (polarized) monochromatic (narrow band) plane wave near resonance to a two-level transition $|g\rangle \to |e\rangle$, for a time T>>1/Γ, where Γ is the atomic linewidth.

(a) Use Fermi's Golden rule to show that

$\sigma_{abs}(\omega_L) = \dfrac{4\pi^2}{\hbar c} \left| \langle e|\hat{\mathbf{d}} \cdot \vec{\varepsilon}_L |g\rangle \right|^2 \omega_L \, g(\omega_L)$,where $g(\omega_L)$ is the normalized atomic lineshape,

and for the case of lifetime broadening, $\sigma_{abs}(\omega_L) = \dfrac{\sigma_0}{1 + 4\Delta^2/\Gamma^2}$, where $\sigma_0 = \dfrac{8\pi\omega_L}{\hbar c} \dfrac{\left| \langle e|\hat{\mathbf{d}} \cdot \vec{\varepsilon}_L |g\rangle \right|^2}{\Gamma}$ is the absorption cross-section on resonance (Δ=0).

(b) Using the expression for the spontaneous emission rate (Einstein-A coefficient), $\Gamma = \dfrac{4}{3\hbar} k_L^3 \left| \langle e|\mathbf{d}|g\rangle \right|^2$, show that the resonant absorption cross section is,

- $\sigma_0 = 6\pi ƛ^2 \left| \langle J_e \, m_g + q | 1 q \, J_g \, m_g \rangle \right|^2$, when the field is polarized along with $\mathbf{e}_q$ and the atom is "polarized", i.e. prepared in a well defined *m*-state.
- $\sigma_0 = 2\pi ƛ^2$ if the field and/or the atom are *unpolarized*.

The bottom line here is that the resonant cross section is of the order of the square of the exciting wave length, **much** larger that the physical cross section of the atom itself for optical resonance.

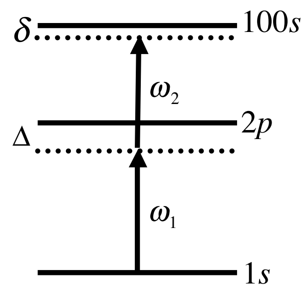
(c) If a photon is absorbed by an atom is will eventually be emitted. This is not necessarily true of other matter, e.g. a molecule or condensed mater (liquid or solid) where the excited matter can relax with other degrees of freedom such as vibrational motion (phonons) etc. The absorption of a photon followed by spontaneous emission is a random direction is *photon scattering*.

Argue that for weak fields, the scattering cross-section is equal to the absorption cross-section, $\sigma_{scatt}(\omega) = \sigma_{abs}(\omega)$, and show that the *scattering-rate* is,

$\gamma_{scatt} = \dfrac{I}{\hbar\omega}\sigma_{scatt} = \dfrac{s}{2}\Gamma$, where $s = \dfrac{\Omega^2/2}{\Delta^2 + \Gamma^2/4}$ is the saturation parameter.

Problem 2: Two-photon transitions (20 points)

Suppose we want to excite a hydrogen atom to a state with a very large principle quantum number *n*=100 (a so-called "Rydberg state"). To get there directly would require an x-ray photon, for which lasers do not exist. If we would like to coherently excite the transition, we must thus resort to a two-photon excitation via a virtual transition. We can enhance the excitation rate by tuning one of the lasers frequencies relatively close to a dipole-allowed strong transition, and then a second photon to connect to the Rydberg state.
For example, suppose we shine two laser beams on the gas with electric fields $\mathbf{E}_1(t) = \mathrm{Re}(\mathbf{E}_1 e^{-i\omega_1 t})$ and $\mathbf{E}_2(t) = \mathrm{Re}(\mathbf{E}_2 e^{-i\omega_2 t})$. We can tune one laser near the $1s \to 2p$ transitions with a detuning Δ, and a second laser so that $\hbar\omega_1 + \hbar\omega_2$ is close to the energy difference between $1s$ and $100s$, with a two-photon detuning of δ, as shown.



(a) Use second order perturbation theory to argue that the transition probability $P_{100s\leftarrow 1s}$ after a time *t* is approximately,

$$P_{100s\leftarrow 1s} = \left(\frac{\Omega_1\Omega_2}{2\Delta\delta}\right)^2 \sin^2(\delta t/2) ,$$

where $\hbar\Omega_1 = \langle 2p|\hat{\mathbf{d}}|1s\rangle \cdot \mathbf{E}_1$ and $\hbar\Omega_2 = \langle 100s|\hat{\mathbf{d}}|2p\rangle \cdot \mathbf{E}_2$ define the Rabi frequencies of the two dipole-allowed transitions.

Now let us solve the problem more exactly according to the time-dependent Schrödinger equation. Restricting the dynamics to these three levels, Hamiltonian in the rotating-wave approximation can be written $\hat{H} = \hat{H}_A + \hat{H}_{\mathrm{int}}(t)$, where

$$\hat{H}_A = E_{1s}|1s\rangle\langle 1s| + E_{2p}|2p\rangle\langle 2p| + E_{100s}|100s\rangle\langle 100s|$$

$$\hat{H}_{\mathrm{int}}(t) = -\frac{\hbar\Omega_1}{2}\left(|2p\rangle\langle 1s|e^{-i\omega_1 t} + |1s\rangle\langle 2p|e^{+i\omega_1 t}\right) - \frac{\hbar\Omega_2}{2}\left(|100s\rangle\langle 2p|e^{-i\omega_2 t} + |2p\rangle\langle 100s|e^{+i\omega_2 t}\right)$$

To simplify notation, let: $|a\rangle = |1s\rangle$, $|b\rangle = |2p\rangle$, $|c\rangle = |100s\rangle$.

(b) In the interaction picture, use the time-dependent Schrödinger equation show that the probability amplitudes in the three states evolve according to

$$\dot{c}_a = i\frac{\Omega_1}{2}e^{+i\Delta t}c_b$$
$$\dot{c}_b = i\frac{\Omega_1}{2}e^{-i\Delta t}c_a + i\frac{\Omega_2}{2}e^{i(\delta-\Delta)t}c_c$$
$$\dot{c}_c = i\frac{\Omega_2}{2}e^{-i(\delta-\Delta)t}c_b$$

(c) In the limit that $|\Delta| >> \Omega_1, \Omega_2, \delta$, the population in the $2p$ state is never large and can be eliminated from the dynamics as a "virtual excitation". To do so, we formally integrate

$$c_b(t) = i\frac{\Omega_1}{2}\int_0^t dt' e^{-i\Delta t'}c_a(t') + i\frac{\Omega_1}{2}\int_0^t dt' e^{i(\delta-\Delta)t'}c_c(t') .$$

Plugging this back in and neglecting the rapidly oscillating terms (which contribute to order $\Omega^3 t/\Delta^2$ and $\Omega^2\delta t/\Delta^2$), show that we get the following approximate equations of motion the couple the probability amplitude in 1s and 100*s* in the two-photon transition,

$$\dot{c}_a = -i\frac{V_a}{\hbar}c_a - i\frac{\Omega_{eff}}{2}c_c e^{i\delta t}$$
$$\dot{c}_c = -i\frac{V_c}{\hbar}c_c - i\frac{\Omega_{eff}}{2}c_a e^{-i\delta t}$$

where $V_a \equiv \frac{\hbar\Omega_1^2}{4\Delta}, V_c \equiv \frac{\hbar\Omega_2^2}{4\Delta}, \Omega_{eff} = \frac{\Omega_1\Omega_2}{2\Delta}$.

(d) With $c_a(0) = 1$, show that up to an overall irrelevant phase, the solution is,

$$c_a(t) = e^{-i\delta t/2}\left(\cos\frac{\tilde{\Omega}t}{2} + i\frac{\delta_{eff}}{\tilde{\Omega}}\sin\frac{\tilde{\Omega}t}{2}\right),\ c_c(t) = ie^{i\delta t/2}\frac{\Omega_{eff}}{\tilde{\Omega}}\sin\frac{\tilde{\Omega}t}{2}$$

where $\delta_{eff} = \delta + (V_c - V_a)/\hbar$ and $\tilde{\Omega} = \sqrt{\Omega_{eff}^2 + \delta_{eff}^2}$

This solution is known as two-photon Rabi flopping.

(e) Please interpret the parameters $\Omega_{eff}, V_a, V_c, \delta_{eff}, \tilde{\Omega}$ (Hint: Compare to the Rabi solution of a two-level resonance).

(f) Show that to lowest order in Ω_1, Ω_2, you recover the solution from perturbation theory in (a)