Lecture 14: Identical Particles and Exchange Symmetry: Bose and Fermi Statistics

Motivation: The postulates of quantum mechanics are incomplete as we have stated them so far. When we deal with a many-body system of identical particles, there are "super selection rules" which limit the possible physical states available.

Examples: Pauli exclusion principle "No two electrons can be in the same state" $\Rightarrow$ Chemistry

- Orbital configurations:
  - H: 1s
  - He: (1s)$^2$
  - Li: (1s)$^2$(2s)
  - Be: (1s)$^2$(2s)$^2$
  - B: (1s)$^2$(2s)$^2$(2p)
  - C: (1s)$^2$(2s)$^2$(2p)$^2$

- Electronics in solids $\Rightarrow$ Fermi energy

Field effect transistor

- Bose-condensation: Quantum Phase Transition

Critical temperature

$$N \frac{\chi^3}{\xi_T} \gg 1$$

$N$: Boson density

$$\xi_T = \sqrt{\frac{1}{4 \kappa m}}$$

Bosons lose "identity" $\Rightarrow$ Condense into single wave function
Identical particles:

- Classical theory - Particles move along well defined trajectories $\Rightarrow$ Even if the particles are identical in every way we can still keep track of which is which.

- Quantum theory - It may be impossible in principle to keep track of particles if identical particles when their wave packets overlap.

  E.g. scattering of identical particles

  \[ \text{Center of mass frame} \]

  \[ \text{Intermediate: particles lose identity} \]

  \[ \text{Final two-particle probability distribution} \]

  \[ \text{Two, in principle, indistinguishable paths} \]

  \[ \text{Feynman's rule: Interfer probability amplitudes for in principle indistinguishable processes} \]
Exchange Symmetry

The description of a many-body system of identical particles fits within the general theory of symmetries.

Exchange symmetry: E.g., two particles

Hamiltonian: $\hat{H}(1, 2) = \hat{H}(2, 1)$ if identical

Define exchange operator (permutation): $\hat{P}_{21}$,

$\hat{P}_{21} \hat{H}(1, 2) \hat{P}_{21}^+ = \hat{H}(2, 1) = \hat{H}(1, 2)$

Symmetry of Hamiltonian

According to our rules so far:

Hilbert space for a given particle $\hat{h}_1 = \hat{h}_2 = \hat{h}$

$\Rightarrow$ Total Hilbert space

$\mathcal{H} = \hat{h}_1 \otimes \hat{h}_2 = \hat{h} \otimes \hat{h}$

Exchange defined: $\hat{P}_{21} \left| \psi_1 \otimes \psi_2 \right> = \left| \psi_1 \right> \otimes \left| \psi_2 \right>$

$\hat{P}_{21}^+ \hat{P}_{21} = \hat{1}$

Given $\hat{S}(1, 2) = \hat{A}(1) \otimes \hat{B}(2)$

$\hat{P}_{21} \hat{S}(1, 2) \hat{P}_{21}^+ = \hat{S}(2, 1) = \hat{B}(1) \otimes \hat{A}(2)$

Hamiltonian is a symmetric operator w.r.t. exchange

$[\hat{H}(1, 2), \hat{P}_{21}] = 0$
Hamiltonian eigenstates are eigenstates of \( \hat{\mathbf{P}}_{21} \).

Note: \( \hat{\mathbf{P}}_{21}^2 = 1 \) (identity)

\( \hat{\mathbf{P}}_{21} \) acts on states:
- \( \hat{\mathbf{P}}_{21} \left| \psi_{\text{S}} \right> = \left| \psi_{\text{S}} \right> \) (Bosons)
- \( \hat{\mathbf{P}}_{21} \left| \psi_{\text{F}} \right> = -\left| \psi_{\text{F}} \right> \) (Fermions)

Connection of exchange theory to intrinsic spin (field theory):
- Elementary particles: \( \frac{1}{2} \) integer (quarks, leptons) \( \Rightarrow \) Fermions
- Whole integer (gauge fields) \( \Rightarrow \) Bosons
- Composite particles \( \leftrightarrow \) Total spin \( \Rightarrow \) Bose or Fermi

Fact of nature: Bosons \( \Rightarrow \) must symmetric \( \Rightarrow \) Physical States
Fermions \( \Rightarrow \) must antisymmetric \( \Rightarrow \) Physical States

Given arbitrary wave function \( \psi(1,2) \), must project onto symmetric/antisymmetric subspaces:

\[
\hat{S} = \frac{1}{2} \left( \hat{1} + \hat{\mathbf{P}}_{21} \right), \quad \hat{A} = \frac{1}{2} \left( \hat{1} - \hat{\mathbf{P}}_{21} \right)
\]

\[
\hat{S} \left| \psi(1,2) \right> = \frac{1}{2} \left( \left| \psi(1,2) \right> + \left| \psi(2,1) \right> \right) \quad \text{(Renormalize)}
\]

\[
\hat{A} \left| \psi(1,2) \right> = \frac{1}{2} \left( \left| \psi(1,2) \right> - \left| \psi(2,1) \right> \right) \quad \text{\( \frac{1}{2} \rightarrow \frac{1}{\sqrt{2}} \)}
\]

For two particles spaces: \( \mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_A \), i.e., \( \hat{S} + \hat{A} = \hat{1} \)

Arbitrary state is spanned by 'H, and \( \mathcal{H}_A \)
Consequences of exchange symmetry

\[ \Rightarrow \text{ Bose and Fermi statistics} \]

Consider two particles with single particle states \( |\phi_A\rangle \) and \( |\phi_B\rangle \).

Possible unsymmetrized states (Maxwell-Boltzmann stats):
\[
|\phi_A\rangle|\phi_A\rangle \quad |\phi_B\rangle|\phi_B\rangle \quad |\phi_A\rangle|\phi_B\rangle \quad |\phi_B\rangle|\phi_A\rangle
\]

Possible symmetric states (Bose-Einstein stats):
\[
|\phi_A\rangle|\phi_A\rangle \quad |\phi_B\rangle|\phi_B\rangle \quad \frac{1}{\sqrt{2}} (|\phi_A\rangle|\phi_B\rangle + |\phi_B\rangle|\phi_A\rangle)
\]

Possible antisymmetric state (Fermi-Dirac stats):
\[
\frac{1}{\sqrt{2}} (|\phi_A\rangle|\phi_B\rangle - |\phi_B\rangle|\phi_A\rangle)
\]

We see that the probability of finding both particles in the same state is:
\[
P_{\text{same}} = \begin{cases} 
\frac{1}{2} & \text{M.B. stats} \\
\frac{2}{3} & \text{B.E. stats} \\
0 & \text{F.D. stats}
\end{cases}
\]

Without symmetrization, there is a 50-50 chance to find particles in the same state. For Bosons, there is an enhancement, and for Fermions, a zero probability. This has important consequences for thermodynamics.
Larger Number of particles: Permutation Group

Given \( N \) particles, there are \( n! \) permutations.

E.g. \( N = 3 \):
- \( \hat{P}_{123} \)
- \( \hat{P}_{231} \)
- \( \hat{P}_{132} \)
- \( \hat{P}_{321} \)

Ex. \( \hat{P}_{132} \) : \( |\psi_A\rangle \otimes |\phi_B\rangle \otimes |\phi_C\rangle = |\phi_A\rangle \otimes |\phi_C\rangle \otimes |\phi_B\rangle \)

Transposition operators \( \hat{P}_{ij} \) (exchange \( i \leftrightarrow j \))

Facts:
- Set of permutations form a group (nonabelian).
- Any permutation is a product of exchanges (not unique).
- The number of exchanges necessary to achieve permutation is unique \( \Rightarrow \) “parity” of permutation \( \Theta = \pm 1 \) (even) \( \Theta = -1 \) (odd).

Projectors:
- \( \hat{S} = \frac{1}{N!} \sum_{\pi \in S_N} \hat{P}_{\pi} \)
- \( \hat{A} = \frac{1}{N!} \sum_{\pi \in S_N} (-1)^{\text{type} \pi} \hat{P}_{\pi} \)

States in subspace \( \mathcal{H}_S \) and \( \mathcal{H}_A \) are eigenstates of \( \hat{P}_{\pi} \)

- \( \hat{P}_{\pi} (|S\psi\rangle) = \chi_\pi \langle S|\psi\rangle \)
- \( \hat{P}_{\pi} (|A\psi\rangle) = (-1)^{(\text{type} \pi)} \langle A|\psi\rangle \)

Symmetry Postulate: Physical states of identical particles lie in \( \mathcal{H}_S \) (Boson) or \( \mathcal{H}_A \) (Fermi).

No superposition of Boson and Fermi states.
Here, for three particles:

\[ \hat{P}_{12} = \hat{P}_{21} \hat{P}_{32} \Rightarrow \sigma = -2 \Rightarrow (-1)^{\sigma} = +1 \text{ even} \]

\[ \hat{P}_{23} \hat{P}_{12} \Rightarrow \sigma = -2 \Rightarrow (-1)^{\sigma} = +1 \text{ (cyclic perm.)} \]

\[ \hat{P}_{13} \hat{P}_{23} \Rightarrow \sigma = 1 \Rightarrow (-1)^{\sigma} = -1 \text{ odd} \]

\[ \hat{P}_{32} \hat{P}_{13} \Rightarrow \sigma = 1 \text{ (noncyclic)} \]

Given three particle state \[ |\psi_A > |\psi_B > |\psi_c > = \sum |\Psi > \]

**Unnormalized:** \[ |\Psi > = \sum_{\frac{3!}{2}} |\psi > = \frac{1}{6} \sum |\psi > \]

**Normalized:** \[ |\Psi > = \frac{1}{\sqrt{6}} \left( |\phi_A > |\phi_B > |\phi_c > + |\phi_c > |\phi_A > |\phi_B > + |\phi_B > |\phi_c > |\phi_A > + \\
+ |\phi_A > |\phi_c > |\phi_B > + |\phi_B > |\phi_A > |\phi_c > + |\phi_c > |\phi_B > |\phi_A > \right) \]

**Unnormalized:** \[ |\psi_A > = \hat{A} |\Psi > = \frac{1}{6} \sum_{\frac{3!}{2}} (-1)^{\sigma(\text{perm})} |\psi > \]

**Normalized:** \[ |\psi_A > = \frac{1}{\sqrt{6}} \left( |\phi_A > |\phi_B > |\phi_c > + |\phi_c > |\phi_A > |\phi_B > + |\phi_B > |\phi_c > |\phi_A > - \\
- |\phi_A > |\phi_c > |\phi_B > - |\phi_B > |\phi_A > |\phi_c > - |\phi_c > |\phi_B > |\phi_A > \right) \]

**Note:** Without exchange symmetry there are 6 possible states with one particle \( |\phi_A > \) another \( |\phi_B > \) and the last \( |\phi_c > \).

Under the symmetrization rule, only two physical states \( |\Psi > \) for Bosons, \( 1^{2}P_{A} \) for Fermions.

**Note:** If any two \( \phi > > s \) are the same \( |\Psi > = 0 \) Pauli Principle
For the Fermionic case, given $N$ particles, with possible single particle states, $\phi_A, \phi_B, \phi_C, \ldots$ we can construct the completely antisymmetric state using a determinant: the Slater determinant.

E.g. $N=3$ which one electron in $\phi_A$ another in $\phi_B$ a third in $\phi_C$ and a fourth in $\phi_C$

$$\Phi(1, 2, 3) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_A(1) & \phi_B(1) & \phi_C(1) \\ \phi_A(2) & \phi_B(2) & \phi_C(2) \\ \phi_A(3) & \phi_B(3) & \phi_C(3) \end{vmatrix}$$

It is relatively straightforward to check that

$$\hat{\sigma}_3 \Phi = (-1)^{\sigma_3(\phi_3)} \Phi$$

Also note that if any two $\phi$'s are the same $\Phi$ vanishes. - Pauli Principle

In contrast for Bosons if say $\phi_C = \phi_A$

$$|123\rangle = \frac{1}{\sqrt{3}} (|\phi_A\rangle|\phi_A\rangle|\phi_B\rangle + |\phi_A\rangle|\phi_B\rangle|\phi_A\rangle + |\phi_B\rangle|\phi_A\rangle|\phi_A\rangle)$$

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