Problem 1: Berry’s Phase (20 points)
We have studied the adiabatic theorem, which roughly states that under evolution according to time-dependent Schrödinger equation (TDSE) with a time-dependent Hamiltonian, if the Hamiltonian changes “slowly enough,” the probability to be in the instantaneous eigenstate is conserved. We shoved one point under the rug – the phase of the adiabatically evolved state. This phase has an important geometry property and is intimately tied to the theory of gauge invariance we have studied.

Consider a Hamiltonian that is a function of a classical set of parameters (a vector) that is a function of time $\hat{H}(\mathbf{R}(t))$. This has instantaneous eigenstates, which we denote

$$\hat{H}(\mathbf{R}(t))|u_n(\mathbf{R}(t))\rangle = E_n(\mathbf{R}(t))|u_n(\mathbf{R}(t))\rangle.$$ 

Suppose at initial time the state is in an instantaneous eigenstate, $|\psi(0)\rangle = |u_n(R(0))\rangle$. The state evolves according to the TDSE: $i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(\mathbf{R}(t))|\psi(t)\rangle$. Under the assumption that the change in $\mathbf{R}(t)$ is adiabatic, we can make the Ansatz for the solution

$$|\psi(t)\rangle = e^{i\gamma_n(t)} e^{-\frac{i}{\hbar} \int_0^t E_n(\mathbf{R}_s(t')) dt'} |u_n(R(t))\rangle.$$ 

The factor $e^{-\frac{i}{\hbar} \int_0^t E_n(\mathbf{R}_s(t')) dt'}$ is arises from so-called the “dynamical phase” expected from the TDSE. The factor $e^{i\gamma_n(t)}$ is additional. $\gamma_n(t)$ is known as Berry’s phase, after Michael Berry who developed the theory, or the geometrical phase for reasons we will see.

(a) Using the Ansatz above and the TDSE under the adiabatic approximation, show that Berry’s phase can be expressed as

$$\gamma_n(t) = \int_{\mathbf{R}(0)}^{\mathbf{R}(t)} d\mathbf{R} \cdot \langle u_n(\mathbf{R})|i\nabla_{\mathbf{R}}|u_n(\mathbf{R})\rangle$$

where $C$ denotes some contour of the path connecting $\mathbf{R}(0) \to \mathbf{R}(t)$.

(b) Berry’s phase has the form of phase associated with the motion of a charged particle in an effective-magnetic field with effective-vector potential
\[\gamma_n(t) = \int_{R(t)}^{R(0)} dR \cdot A_{\text{eff}}(R), \text{ where } A_{\text{eff}}(R) = \langle u_n(R) | i \nabla_R | u_n(R) \rangle\]

This is a beautiful result. In particular, suppose \( R \) is change in a closed path, \( R(t) = R(0) \).

Show that \( \gamma_n(t) = \int_S d\mathbf{a} \cdot \mathbf{B}_{\text{eff}}(R) \), where \( S \) is a surface enclosed by path, and

\[
\mathbf{B}_{\text{eff}} = \nabla \times \mathbf{A}_{\text{eff}} = -\text{Im} \left( \left( \nabla_R \langle u_n(R) | \times \nabla_R | u_n(R) \rangle \right) \right)
\]

\[= -\text{Im} \left( \sum_{m \neq n} \frac{\langle u_n(R) | \nabla_R H(R) | u_m(R) \rangle \times \langle u_m(R) | \nabla_R H(R) | u_n(R) \rangle}{(E_m(R) - E_n(R))^2} \right)\]

A key take away message is that Berry’s phase is geometrical. For a closed path through parameter space, the phase depends only on the area of the enclosed path, and not the specific path taken.

(c) As an example, let us return to a problem we’ve seen before, evolution of a spin-1/2 particle in true magnetic field whose direction is changed as a function of time, \( \mathbf{B}(t) = B(t) \hat{\mathbf{z}} \). The time-dependent Hamiltonian is \( \hat{H}(\mathbf{B}(t)) = \mathbf{B}(t) \cdot \hat{\mathbf{\sigma}} \), where \( \hbar \Omega = \mu_B B \). The field \( \mathbf{B}(t) \) plays the role of the classical parameter \( \mathbf{R}(t) \) above.

Suppose at \( t=0 \) the start the true magnetic field along the \( z \)-axis and the spin is up.

Show that under Berry’s effective \( B \)-field is \( B_{\text{eff}}(\mathbf{B}(t)) = -\frac{\mathbf{b}(t)}{2B^2} \). In the parameter space \( \mathbf{R}(t) = \mathbf{B}(t) \), this is the magnetic field of a magnetic monopole.

Hint: Do this for \( \mathbf{b} = \mathbf{e}_z \) and then generalize.

(d) Show that under a closed loop Berry’s phase is \( \gamma = \pm \frac{\Omega}{2} \), where \( \Omega \) is the solid angle swept out but the tip of the true magnetic field, and \( \pm \) depends on the direction of the loop. This is purely geometrical.
Problem 2: Hydrogenic atoms and atomic units. (10 points)

Consider the “hydrogenic” atoms - that is bound-states of two oppositely charged particles:

(i) The hydrogen atom: Binding of an electron and proton.
(ii) Heavy ion: Single electron bound to a nucleus of mass $M$, charge $Ze$ (say $Z=50$).
(iii) Muonium: Muon bound to a proton
(iv) Positronium: Bound state of an electron and a positron (anti-electron)

(a) For each, using the charges, reduced mass, and the unit $\hbar$, determine the characteristic scales of: Length, energy, time, momentum, and internal electric field. Please give numerical values as well as the expressions in terms of the fundamental constants.

(b) Now add the speed of light $c$ into the mix. Find characteristic velocity in units of $c$, magnetic field, and magnetic moment. Show that for the particular case of hydrogen the characteristic velocity is $v/c = \alpha = \frac{e^2}{\hbar c}$ (cgs) $\approx 1/137$, the “fine-structure” constant, and that the Bohr radius, Compton wavelength, and “classical electron radius”, differ by powers of $\alpha$ according to,

$$r_{\text{class}} = \alpha \lambda_{\text{compton}} = \alpha^2 a_0$$