

Physics 522: Quantum II

Lecture 2: Review: From Hilbert Space to Physical Space

As Asher Peres once said, "Quantum Mechanics happens in the lab, not in Hilbert Space!" So far our discussion has been completely abstract, vectors in Hilbert space and linear operators on the space. The connection to physical quantities is symmetry. The physical world can exhibit certain symmetry, transformations that leave the laws of physics unchanged. The key connection to physics is

Noether Theorem: To each (continuous) symmetry of a physical system there is a conserved quantity that is the "generator" of that symmetry.

We will explain what this means shortly, but the key idea is that symmetries in the physical world are associated with physical quantities.

How then to connect this to quantum mechanics? A symmetry is a map on Hilbert space that leaves the physics invariant. What this means then, is that if everything is transformed according to the symmetry, then the measurement predictions, i.e. probability of a measurement outcome occurring, must be preserved under a symmetry map. This leads us to

Wigner Theorem: Symmetry maps in quantum mechanics are implemented by unitary (or anti-unitary) operator

Suppose \hat{S} is symmetry operation. Let $\hat{S}|\psi\rangle = |\tilde{\psi}\rangle \quad \forall |\psi\rangle \in \mathcal{H}$. Then according we require $P(\phi|\psi) = P(\tilde{\phi}|\tilde{\psi})$. According to the Born rule, we must then have

$$|\langle \phi | \psi \rangle|^2 = |\langle \tilde{\phi} | \tilde{\psi} \rangle|^2.$$

$$\Rightarrow |\langle \phi | \psi \rangle| = |\langle \phi | \hat{S}^\dagger \hat{S} | \psi \rangle| \Rightarrow \hat{S} \text{ is unitary or anti-unitary}$$

Note: Clearly if $\hat{S}^\dagger \hat{S} = \hat{1}$, i.e. \hat{S} is unitary, Wigner's theorem is satisfied.

An anti-Unitary operator is one such that $\langle \psi | \hat{S}^\dagger \hat{S} | \phi \rangle = \langle \psi | \phi \rangle^* = \langle \phi | \psi \rangle$.

Note, both unitary anti-unitary operators preserve the magnitude of the inner product, and thus satisfy Wigner's theorem.

Of particular importance are continuous symmetries, i.e., symmetry transformations that are parameterized by a continuous real variable. Examples:

- Spatial translation: $\hat{T}(x)$
- Momentum translation: $\hat{M}(p)$
- Rotation around an axis \vec{n} : $\hat{R}_{\vec{n}}(\theta)$
- Time translation: $\hat{U}(t)$

Generally, for all these symmetries parameterized by a real variable λ , $\hat{S}(\lambda)$, as $\lambda \rightarrow 0$, $\hat{S} \rightarrow \hat{1} \Rightarrow$ Continuous symmetries are unitary. Since this is continuous, we must have for $\lambda \ll 1$

$$\lim_{\lambda \rightarrow 0} \hat{S}(\lambda) = \hat{1} + i\lambda \hat{A}, \quad \text{where } \hat{A} = \hat{A}^\dagger \text{ (Hermitian)}$$

Proof

$$\hat{S}^\dagger \hat{S} = \hat{1} = \hat{1} + i\lambda (\hat{A} - \hat{A}^\dagger) = \hat{1}$$

As λ becomes finite, $\hat{S}(\lambda) = e^{i\lambda \hat{A}}$. This is an example of a Lie group in 1D, as λ is along a one-dimensional line. The Hermitian operator \hat{A} is known as the generator of the group. According to Noether's theorem, the generator of the symmetry is proportional to the corresponding physically conserved quantity.

- Spatial translation symmetry \iff Momentum is conserved
- Momentum translation symmetry \iff Position is conserved
- Rotational symmetry about an axis \iff Component of angular momentum conserved
- Time translation symmetry \iff Energy is conserved

- ⇒ Momentum is generator of spatial translation: $\hat{T}(x) = e^{-\frac{i x \hat{p}}{\hbar}}$, \hat{p} = Momentum operator
- ⇒ Position is generator of momentum translation: $\hat{M}(p) = e^{+i p \hat{x}}/\hbar$, \hat{x} = position operator
- ⇒ Angular momentum component is generator of rotation: $\hat{R}_n(\theta) = e^{-i \theta \frac{\hat{L}_n}{\hbar}}$, \hat{L} = angular momentum op.
- ⇒ Energy is the generator of time translation: $\hat{U}(t) = e^{-\frac{i t \hat{H}}{\hbar}}$, \hat{H} = Hamiltonian = Energy op.

Note: We have introduced here "h-bar", \hbar , which is the fundamental unit of "action."
 It relates physical resource - energy, time, position, momentum - to distinguishable state in Hilbert space. We thus see that these physical symmetries connect Hilbert space of physical observables.

Hilbert space for a particle moving in 1D

In classical physics, a particle moving along a line is described by a phase space (x, p) .
 In quantum physics we have operator \hat{x} and \hat{p} . The eigenvectors have continuous eigenvalues:

$$\hat{x}|x\rangle = x|x\rangle, \quad \hat{p}|p\rangle = p|p\rangle$$

The resolution of the identity is thus an integral

$$\hat{1} = \int_{-\infty}^{\infty} dx |x\rangle\langle x| = \int_{-\infty}^{\infty} dp |p\rangle\langle p|$$

The representation of states in Hilbert space are functions of a real number

$$|\psi\rangle \in \mathcal{H} \Leftrightarrow \psi(x) = \langle x|\psi\rangle \text{ (position space wave function)}$$

$$\tilde{\psi}(p) = \langle p|\psi\rangle \text{ (momentum space wave function)}$$

$$\langle \psi|\phi\rangle = \int dx \langle \psi|x\rangle\langle x|\phi\rangle = \int dx \psi^*(x) \phi(x) = \int dp \tilde{\psi}^*(p) \tilde{\phi}(p)$$

$$\|\psi\|^2 = \langle \psi|\psi\rangle = \int dx |\psi(x)|^2 = \int dp |\tilde{\psi}(p)|^2$$

⇒ $\mathcal{H} = \mathcal{L}_2(\mathbb{R}) =$ Square normalizable functions on the real line.

This Hilbert space is ∞ -dimensional.

Note: $|x\rangle$ and $|p\rangle$ are actually not in \mathcal{H} ; they are not square normalizable.

$$\text{We have } \langle x|x'\rangle = \delta(x-x'), \quad \langle p|p'\rangle = \delta(p-p') \Rightarrow \|x\| = \|p\| = \infty$$

But they are in the dual space, as $\langle x|\psi\rangle = \psi(x)$ is finite.

The Schrödinger Equation

Kinematics and dynamics is central to our description of physics. Central to this is the time translation operator:

$$\hat{U}(t) = e^{-\frac{i\hat{H}t}{\hbar}} \quad : \quad \hat{H} = \text{Hamiltonian}$$

That is given the initial state at time $t=0$ $|\psi(0)\rangle$ the state at a later time is

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$$

We can consider $\hat{U}(t)$ as the solution to the following differential equation:

$$\frac{\partial}{\partial t} \hat{U}(t) = -\frac{i}{\hbar} \hat{H} \hat{U}(t)$$

$$\text{Thus } \frac{\partial}{\partial t} |\psi(t)\rangle = -\frac{i}{\hbar} \hat{H} |\psi(t)\rangle$$

These equations are known as the time-dependent Schrödinger equation T.D.S.E

Of particular interest are the eigenstates of the Hamiltonian, i.e. satisfying

$$\hat{H} |u_E\rangle = E |u_E\rangle$$

This is known as the time-independent Schrödinger equation. T.I.S.E.

The importance of the energy eigenstates is that they are stationary states. That is, if the system is in an energy eigenstate, then for all times the probability of any measurement is independent of time (stationary). To see this, note

$$\text{If } |\psi(0)\rangle = |u_E\rangle, \text{ then } |\psi(t)\rangle = \hat{U}(t)|u_E\rangle = e^{-\frac{i\hat{H}t}{\hbar}} |u_E\rangle$$

$$\Rightarrow |\psi(t)\rangle = e^{-\frac{iEt}{\hbar}} |u_E\rangle$$

The overall phase does not change any predicted measurement probabilities
 \Rightarrow Stationary State

The stationary state are the kinematic components in quantum mechanics. They determine the energy levels, between which dynamics can occur. Solving the T.I.S.E. is thus an important part of physics that defines the structure of matter at all scales from the subatomic to atoms, molecules, and condensed matter.

The solution to the T.I.S.E. determines the solution to the T.D.S.E. Suppose at time $t=0$, the state is $|\psi(0)\rangle$. Because the energy eigenstates form an basis for Hilbert space

$$|\psi(0)\rangle = \sum_E c_E |\psi_E\rangle, \quad \text{where } c_E = \langle \psi_E | \psi(0) \rangle$$

The state at a later time is then

$$|\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle = \sum_E c_E \hat{U}(t) |\psi_E\rangle = \sum_E \underbrace{c_E e^{-iEt/\hbar}}_{c_E(t)} |\psi_E\rangle$$

thus we have a solution to the T.D.S.E.