

Physics 522: Quantum II

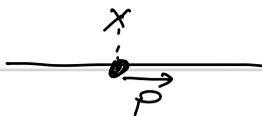
Lecture 3: Quantum Mechanics of multiple degrees of freedom

Multiple Degrees of Freedom: Classical Physics

In classical physics, each "degree of freedom" is assigned a pair of canonical coordinates (x, p) . For Example:

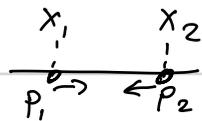
One degree of freedom

- Particle moving along real lines

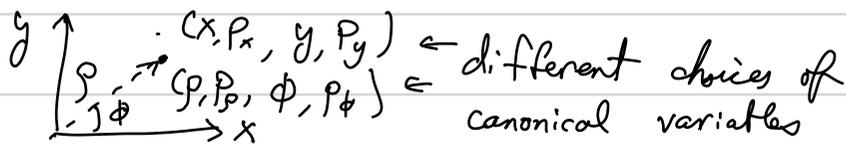


Two degrees of freedom

- Two particles moving along real line



- One particle moving in a 2D plane



Generally, given n degrees of freedom, the joint phase space is the Cartesian product of the phase spaces for each degree of freedom

$$(x_1, p_1, x_2, p_2, \dots, x_n, p_n) = (x_1, p_1) \times (x_2, p_2) \times \dots \times (x_n, p_n)$$

2n-dimensional phase space

Quantum Mechanics

One degree of freedom: Position/Momentum operators $[\hat{x}, \hat{p}] = i\hbar$

Hilbert space \mathcal{H} , defined by position/momentum representations

$$|\psi\rangle \in \mathcal{H} \text{ if } \langle \psi | \psi \rangle < \infty \Rightarrow \int_{-\infty}^{\infty} dx |\psi(x)|^2 = \int_{-\infty}^{\infty} dp |\tilde{\psi}(p)|^2 < \infty$$

$\Rightarrow \mathcal{H} = \mathcal{L}_2(\mathbb{R}) =$ square normalizable functions on a line
: infinite dimensional

$$\text{Basis } \{|u_\alpha\rangle\} \quad \sum_{\alpha=1}^{\infty} |u_\alpha\rangle \langle u_\alpha| = 1 \quad \Rightarrow \quad \sum_{\alpha=1}^{\infty} u_\alpha(x) u_\alpha^*(x') = \delta(x-x')$$

"Completeness"

$$\sum_{\alpha=1}^{\infty} \langle x | u_\alpha \rangle \langle u_\alpha | x' \rangle = \langle x | x' \rangle$$

Two degrees of freedom

E.g. particle moving in 2D. Canonical observables $\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y$

Note \hat{p}_x : Generator of spatial translation along x

\hat{p}_y : " " " " " "

These can be done either order to get to same position $\Rightarrow [\hat{p}_x, \hat{p}_y] = 0$

Similarly $[\hat{x}, \hat{y}] = 0$ (momentum translations in either order commute).

Final momentum translation along x commutes with spatial translation along y and vice versa $\Rightarrow [\hat{x}, \hat{p}_y] = [\hat{y}, \hat{p}_x] = 0$

\Rightarrow Canonical commutators $[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0$ $[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$

Define Hilbert space in position representation: $\mathcal{H} = \mathcal{L}_2(\mathbb{R}_2)$

Wave function over plane: $\{\Psi(x, y)\}$ $\int dx dy |\Psi(x, y)|^2 < \infty$:

Claim: If $\{u_\alpha(x)\}$ is a basis for $\mathcal{L}_2(\mathbb{R})$ along x , and $\{v_\beta(y)\}$ a basis for $\mathcal{L}_2(\mathbb{R})$ along y the $\{W_{\alpha\beta}(x, y) = u_\alpha(x) v_\beta(y)\}$ is a basis for $\mathcal{L}_2(\mathbb{R}_2)$

Proof: A basis of functions must be "complete", i.e. span the space. If normalized

$$\sum_{\alpha, \beta} V_{\alpha\beta}(x, y) V_{\alpha\beta}^*(x', y') = \delta^{(2)}(\vec{x} - \vec{x}') = \delta(x - x') \delta(y - y')$$

$$\text{(here)} = \left(\sum_{\alpha} u_{\alpha}(x) u_{\alpha}^*(x') \right) \left(\sum_{\beta} v_{\beta}(y) v_{\beta}^*(y') \right) = \delta(x - x') \delta(y - y')$$

The Hilbert space formed by the span of all product basis vectors is known as the tensor product Hilbert space of the two Hilbert spaces

$$\mathcal{L}_2(\mathbb{R}_2) = \mathcal{L}_2(\mathbb{R}) \otimes \mathcal{L}_2(\mathbb{R})$$

$$\mathcal{H}_{x,y} = \mathcal{H}_x \otimes \mathcal{H}_y$$

$$\Rightarrow \forall \Psi(x, y) \in \mathcal{H}_{x,y} \quad \Psi(x, y) = \sum_{\alpha, \beta} c_{\alpha\beta} u_{\alpha}(x) v_{\beta}(y)$$

Similarly, two particles moving one real line: $\mathcal{H}_{1,2} = \mathcal{H}_1 \otimes \mathcal{H}_2$

Consider then a generic Hilbert space for two (spatial) degrees of freedom, $\mathcal{H}_{1,2} = \mathcal{H}_1 \otimes \mathcal{H}_2$. It is clear that if $\psi_1(x_1) \in \mathcal{H}_1$, and $\psi_2(x_2) \in \mathcal{H}_2$ then
$$\underline{\Psi}(x_1, x_2) = \psi_1(x_1) \psi_2(x_2) \in \mathcal{H}_{1,2}$$

This is known as a "product state" or a "separable state".

For such a state, the joint probability density for finding the particle near x_1 and x_2
$$P(x_1, x_2) = |\underline{\Psi}(x_1, x_2)|^2 = |\psi_1(x_1)|^2 |\psi_2(x_2)|^2 = P_1(x_1) P_2(x_2).$$

In statistics, the two random variables x_1 and x_2 are said to be uncorrelated. That is, the probability of finding x_1 (in interval $x_1 \rightarrow x_1 + dx_1$) is independent of the value of x_2 , and vice versa.

We define the marginal probability density as in statistics

$$P_1(x_1) = \int_{-\infty}^{\infty} dx_2 P(x_1, x_2) = \text{Probability of finding } x_1, \text{ irrespective value of } x_2$$

$$P_2(x_2) = \int_{-\infty}^{\infty} dx_1 P(x_1, x_2)$$

According to rules of probability
$$P(x_1, x_2) = \underbrace{P(x_1 | x_2)}_{\text{conditional probability density}} P_2(x_2) = \underbrace{P(x_2 | x_1)}_{\text{conditional probability density}} P_1(x_1)$$

For uncorrelated events $P(x_1 | x_2) = P_1(x_1)$, , , $P(x_2 | x_1) = P_2(x_2)$

The separable pure state is such that we assign a separate wave function to each degree of freedom: $P_1(x_1) = |\psi_1(x_1)|^2$, $P_2(x_2) = |\psi_2(x_2)|^2$.

We can use these wave functions to define any (compatible) measurements

e.g. $P(p_1, p_2) = |\tilde{\Psi}_1(p_1)|^2 |\tilde{\Psi}_2(p_2)|^2$ where $\tilde{\Psi}_1(p_1) = \int \frac{dx_1}{\sqrt{2\pi}} e^{ip_1 x_1 / \hbar} \psi_1(x_1)$, etc.

$$P(x_1, p_2) = |\psi_1(x_1)|^2 |\tilde{\Psi}_2(p_2)|^2, \quad P(p_1, x_2) = |\tilde{\Psi}_1(p_1)|^2 |\psi_2(x_2)|^2$$

Note \hat{x}_1, \hat{p}_2 are compatible variables, so we can assign a joint probability distribution to $P(x_1, p_2)$ and similarly for \hat{p}_1, \hat{x}_2

Entangled States

Not every pure state $\Psi(x_1, x_2) \in \mathcal{H}_{1,2}$ is a product state. For example. Suppose $u(x) \perp v(x)$ are orthogonal functions

$$\text{Let } \Psi(x_1, x_2) = \frac{1}{\sqrt{2}} u(x_1) v(x_2) + \frac{1}{\sqrt{2}} v(x_1) u(x_2) \neq \psi(x_1) \phi(x_2) \\ \text{for any } \psi \perp \phi$$

Such a state is an example of an entangled state. One characteristic of an entangled state is that the measurement outcomes are correlated.

$$P(x_1, x_2) = |\Psi(x_1, x_2)|^2 = \frac{1}{2} |u(x_1)|^2 |v(x_2)|^2 + \frac{1}{2} |v(x_1)|^2 |u(x_2)|^2 \\ + \frac{1}{2} (u^*(x_1) v(x_1) u(x_2) v^*(x_2) + \text{c.c.}) \\ \neq P_1(x_1) P_2(x_2) \text{ for any } P_1 \perp P_2$$

While the entangled state shows correlated outcomes, in fact, these correlations cannot be described in terms of classical conditional probability distributions. In fact the correlations are in some sense "stronger." This leads to the famous Bell's inequalities the bounds the classical correlations when compared with quantum. One indicator of the unique aspect of entanglement is the nature of the marginal distribution

$$P_1(x_1) = \int dx_2 P(x_1, x_2) = \frac{1}{2} |u(x_1)|^2 \int dx_2 |v(x_2)|^2 + \frac{1}{2} |v(x_1)|^2 \int dx_2 |u(x_2)|^2 \\ + \frac{1}{2} (u^*(x_1) v(x_1) \underbrace{\int dx_2 u(x_2) v^*(x_2)}_{\langle v|u \rangle = 0} + \text{c.c.})$$

$$\Rightarrow P_1(x_1) = \frac{1}{2} |u(x_1)|^2 + \frac{1}{2} |v(x_1)|^2$$

Similarly, $P_2(x_2) = \frac{1}{2} |u(x_2)|^2 + \frac{1}{2} |v(x_2)|^2$. This is a very odd state of affairs. The probability of finding particle-1 @ x_1 is a statistical mixture of the two probability distributions $|u(x)|^2$ and $|v(x)|^2$ - there is no quantum interference between them (the same is true of $P_2(x_2)$). There is no pure state we can assign to particle 1 or particle 2 alone, even though the overall joint state is pure $\Psi(x_1, x_2)$. This is the essence of entanglement. We can have maximal possible knowledge of the whole but incomplete knowledge of the parts. This is "quantum weirdness."

General rules of the tensor product

The abstraction follows from those things that work for products of functions of different variables.

$$\text{Let } \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \quad \begin{array}{l} |\psi\rangle_A \in \mathcal{H}_A \\ |\phi\rangle_B \in \mathcal{H}_B \end{array}$$

Linearly \Rightarrow

$$\begin{cases} (1) (c|\psi\rangle_A) \otimes |\phi\rangle_B = |\psi\rangle_A \otimes (c|\phi\rangle_B) = c(|\psi\rangle_A \otimes |\phi\rangle_B) \\ (2) |\psi\rangle_A \otimes (|\phi_1\rangle_B + |\phi_2\rangle_B) = |\psi\rangle_A \otimes |\phi_1\rangle_B + |\psi\rangle_A \otimes |\phi_2\rangle_B \end{cases}$$

~~Let~~ Let $|\Psi_1\rangle = |\psi_1\rangle_A \otimes |\phi_1\rangle_B$
 $|\Psi_2\rangle = |\psi_2\rangle_A \otimes |\phi_2\rangle_B$

Inner and Outer product \Rightarrow

$$(3) \langle \Psi_1 | \Psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle_A \langle \phi_1 | \phi_2 \rangle_B$$

$$(4) |\Psi_1\rangle \langle \Psi_2| = |\psi_1\rangle_A \langle \psi_2| \otimes |\phi_1\rangle_B \langle \phi_2|$$

$$\text{Let } \hat{O} = \hat{A} \otimes \hat{B}$$

$$\Rightarrow (5) \hat{O} |\Psi\rangle = \hat{A} |\psi\rangle_A \otimes \hat{B} |\phi\rangle_B$$

Operators

$$(6) \hat{O}_1 \hat{O}_2 = (\hat{A}_1 \hat{A}_2) \otimes (\hat{B}_1 \hat{B}_2)$$

Matrix manipulation

Consider $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ where

\mathcal{H}_A is spanned by $\{ |e_i\rangle_A \mid i=1, 2, \dots, d_A \}$

\mathcal{H}_B " " " $\{ |f_j\rangle_B \mid j=1, 2, \dots, d_B \}$

$\Rightarrow \mathcal{H}_{AB}$ " " " $\{ |e_i\rangle_A \otimes |f_j\rangle_B \}$

We take a standard orders of the product states

$$|e_1\rangle_A \otimes |f_1\rangle_B$$

$$|e_1\rangle_A \otimes |f_2\rangle_B$$

⋮

$$|e_1\rangle_A \otimes |f_{d_B}\rangle_B$$

$$|e_2\rangle_A \otimes |f_1\rangle_B$$

⋮

$$|e_2\rangle_A \otimes |f_{d_B}\rangle_B$$

⋮

$$|e_{d_A}\rangle_A \otimes |f_1\rangle_B$$

With this ordering we can look at matrix representations on \mathcal{H}_{AB} and relate them to matrix representation of tensor products

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Example: Consider $d_A = 2$ $d_B = 2$

Standard order \Rightarrow

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using this and rules (1)+(2) of the tensor product

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \otimes \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 v_1 \\ u_1 v_2 \\ u_2 v_1 \\ u_2 v_2 \end{bmatrix} = \begin{bmatrix} u_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ u_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{bmatrix}$$

In other words, there are two "blocks".

Now for operators (matrices)

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\text{outer product})$$

$$\begin{aligned} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= (\begin{bmatrix} 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix}) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \quad \text{rule (4)} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

etc. for all 16 matrices with one entry 1 and all others 0

Again by linearity,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{21} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ a_{22} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{22} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{bmatrix} \leftarrow \text{Blocks}$$

Examples:

$$\hat{\sigma}_x \otimes \mathbb{1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{1} \otimes \hat{\sigma}_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\hat{\sigma}_z \otimes \hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: If $h =$ space of dim d

$$\underbrace{h \otimes h \otimes \dots \otimes h}_{N \text{ times}} = h^{\otimes N} = \text{dimension } d^N$$

\Rightarrow Space exponentially bigger! This is

unique to quantum mechanics, this complexity may be a resource in quantum information processing.

Joint states and Marginal States

Consider first the classical case of two random variables r_1 and r_2 . We can write ~~the~~ a joint probability distribution $P(r_1, r_2)$ such that, e.g.

$$\overline{f(r_1, r_2)} = \sum_{r_1, r_2} f(r_1, r_2) P(r_1, r_2)$$

for any function of the two variables.

Suppose our function does not depend on r_2

$$\begin{aligned} \overline{f(r_1)} &= \sum_{r_1, r_2} f(r_1) P(r_1, r_2) = \sum_{r_1} f(r_1) \sum_{r_2} P(r_1, r_2) \\ &\equiv \sum_{r_1} f(r_1) p(r_1) \end{aligned}$$

Where $p(r_1) = \sum_{r_2} P(r_1, r_2)$ is the marginal probability distribution. The marginal

determines the statistics of variable r_1 , irrespective of what value r_2 takes on.

Note: If r_1 and r_2 are statistically independent

$$P(r_1, r_2) = p_1(r_1) p_2(r_2)$$

⇒ Marginals are obvious

If $P(r_1, r_2)$ cannot be factorized, r_1 and r_2 are correlated (in a classical statistical sense).

What about quantum mechanically?

We consider a joint space of two degrees of freedom

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Any state ~~state~~ on \mathcal{H}_{AB} is a density operator $\hat{\rho}_{AB}$

Then, given operator \hat{O}_{AB} acting on \mathcal{H}_{AB}

$$\begin{aligned}\langle \hat{O}_{AB} \rangle &= \text{Tr}_{AB} (\hat{O}_{AB} \hat{\rho}_{AB}) \\ &= \sum_{ij} \langle e_i f_j | \hat{O}_{AB} \hat{\rho}_{AB} | e_i f_j \rangle\end{aligned}$$

$$\text{where } |e_i f_j\rangle \equiv |e_i\rangle_A \otimes |f_j\rangle_B$$

Now suppose $\hat{O}_{AB} = \hat{A} \otimes \hat{\mathbb{1}}_B$ (i.e. operator only on \mathcal{H}_A)

$$\Rightarrow \langle \hat{A} \otimes \hat{\mathbb{1}}_B \rangle = \sum_{ij} \langle e_i f_j | \hat{A} \otimes \hat{\mathbb{1}}_B \hat{\rho}_{AB} | e_i f_j \rangle$$

$$= \sum_i \langle e_i | \hat{A} \sum_j \langle f_j | \hat{\rho}_{AB} | f_j \rangle | e_i \rangle$$

$$\Rightarrow \langle \hat{A} \rangle \equiv \sum_i \langle e_i | \hat{A} \hat{\rho}_A | e_i \rangle = \text{Tr}_A (\hat{A} \hat{\rho}_A)$$

where $\hat{\rho}_A = \text{Tr}_B (\hat{\rho}_{AB})$ is the marginal or reduced density operator
↑
"partial trace"

Entanglement and purity of marginal state

Let us restrict our attention to joint states which are pure

$$\Rightarrow \hat{\rho}_{AB} = |\Psi\rangle_{AB} \langle\Psi|_{AB}$$

Recall: $|\Psi\rangle_{AB}$ is separable iff $\exists |\psi\rangle_A$ and $|\phi\rangle_B$ s.t. $|\Psi\rangle_{AB} = |\psi\rangle_A \otimes |\phi\rangle_B$,
otherwise $|\Psi\rangle_{AB}$ is entangled.

$$\text{If } |\Psi\rangle_{AB} \text{ is } \underline{\text{separable}} \Rightarrow \begin{aligned} \hat{\rho}_A &= |\psi\rangle_A \langle\psi| \\ \hat{\rho}_B &= |\phi\rangle_B \langle\phi| \end{aligned} \quad \begin{array}{l} \text{Pure} \\ \text{states} \end{array}$$

$$\Rightarrow \hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B$$

$$|\Psi\rangle_{AB} \text{ separable} \iff \begin{array}{l} \text{(i) marginals are pure} \\ \text{(ii) state is uncorrelated} \end{array}$$

$$\text{If } |\Psi\rangle_{AB} \text{ is } \underline{\text{entangled}} \Rightarrow \hat{\rho}_A \text{ and } \hat{\rho}_B \text{ are } \underline{\text{mixed}}$$

This is a CRUCIAL property of quantum mechanics that distinguishes it from and classical statistical theory. The parts do not define the whole when the system is entangled.

Example: Two spins

$$(1) \quad |\Psi_{AB}\rangle = \frac{1}{2} (|++\rangle + |+-\rangle + |-+\rangle + |--\rangle) \\ = \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}} \right)$$

$\Rightarrow |\Psi_{AB}\rangle$ is separable. In fact $|\Psi_{AB}\rangle = |+\rangle_x \otimes |+\rangle_x$
(spin up along x for both)

\Rightarrow Marginals $\hat{\rho}_A = \hat{\rho}_B = |+\rangle_x \langle +|_x$ Pure

$$(2) \quad |\Psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|+-\rangle - |-+\rangle) \quad \text{entangled}$$

$$\hat{\rho}_{AB} = \frac{1}{2} \left(|+-\rangle \langle +-| - |-+\rangle \langle -+| - |-+\rangle \langle +-| + |+-\rangle \langle -+| \right) \\ = \frac{1}{2} \left[|+\rangle_A \langle +| \otimes |-\rangle_B \langle -| - |+\rangle_A \langle -| \otimes |-\rangle_B \langle +| - |-\rangle_A \langle +| \otimes |+\rangle_B \langle +| \right. \\ \left. + |-\rangle_A \langle -| \otimes |+\rangle_B \langle +| \right]$$

Partial trace

$$\hat{\rho}_A = \text{Tr}_B (\hat{\rho}_{AB}) = \frac{1}{2} |+\rangle_A \langle +| \langle -| - \rangle_B \\ - \frac{1}{2} |+\rangle_A \langle -| \langle +| - \rangle_B \\ - \frac{1}{2} |-\rangle_A \langle +| \langle -| + \rangle_B \\ + \frac{1}{2} |-\rangle_A \langle -| \langle +| + \rangle_B$$

$$\Rightarrow \hat{\rho}_A = \frac{1}{2} |+\rangle_A \langle +| + \frac{1}{2} |-\rangle_A \langle -| = \frac{1}{2} \hat{1}_A$$

$$\text{Similarly } \hat{\rho}_B = \frac{1}{2} |+\rangle_B \langle +| + \frac{1}{2} |-\rangle_B \langle -| = \frac{1}{2} \hat{1}_B$$

This is an extremely odd state of ~~affairs~~ affairs. The state $|\Psi\rangle_{AB}$ is pure

i.e. it constitute maximum possible information about the joint quantum system.

However the marginals are maximally mixed.

\Rightarrow We have no information at all about the individual spins

The fact that there is information in the joint quantum state that cannot be extracted from the constituent pieces is one of the great mysterious features of the theory. It is behind the famous Einstein - Podolsky - Rosen paradox, and Bell's inequalities, as well as the source of new information processing protocols.