Lecture 5 : Angular Momentum Algebra

Rotation symmetry

• Symmetry review

We have learned so far about three continuous symmetries: time translation, spatial translation, and momentum translation. Using Noether's theorem, we find that each symmetry ~~[illegible]~~ is associated with corresponding conserved quantity. In quantum mechanics, the symmetries are unitary operators generated by Hermitian operators corresponding to the physical conserved quantities

Time translation: $\underbrace{\hat{U}(\delta t) = 1 - \frac{i\delta t}{\hbar}\hat{H}}_{\text{infinitesimal}} \Rightarrow \underbrace{\hat{U}(t) = e^{-it\hat{H}/\hbar}}_{\text{finite}}$

Spatial translation: $\hat{T}(\delta x) = 1 - \frac{i\delta x}{\hbar}\hat{p} \Rightarrow \hat{T}(x) = e^{-ix\hat{p}/\hbar}$

Momentum translation: $\hat{M}(\delta p) = 1 + \frac{i\delta p}{\hbar}\hat{x} \Rightarrow \hat{M}(p) = e^{+ip\hat{x}/\hbar}$

The latter two can be combined into the phase-space displacement

$$\hat{D}(x_0,p_0) = e^{-\frac{i}{\hbar}(x_0\hat{p} - p_0\hat{x})}$$

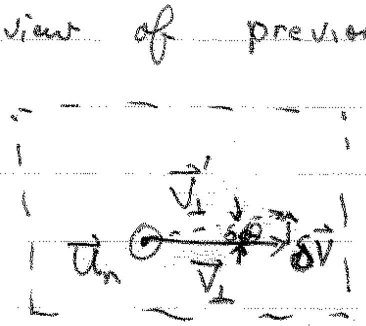
In quantum theory, phase-space displacements generally do not commute

$$\left[\hat{D}(x_0,p_0), \hat{D}(x_1,p_1)\right] = i\sin\left(\frac{x_0p_1 - p_1x_0}{\hbar}\right)\hat{D}(x_0+x_1,\ p_0+p_1)$$

Or infinitesimally $[\hat{x},\hat{p}] = i\hbar$

Infinitesimal rotation

Top view of previous figure



$\delta\theta$ = infinitesimal angle

$$\delta\vec{V} = \delta\theta\,(\vec{u}_n \times \vec{V})$$

$$\Rightarrow \vec{V}' = \vec{V} + \delta\vec{V}$$

$$= \vec{V} + \delta\theta(\vec{u}_n \times \vec{V})$$

Noether theorem: System which is invariant under rotations $\Rightarrow$ Angular momentum is conserved

Classically $\vec{L} = \vec{x} \times \vec{p}$ (orbital angular momentum

Quantum Mechanically:

Two kinds: orbital $\hat{\vec{L}} = \hat{\vec{x}} \times \hat{\vec{p}}$

spin $\hat{\vec{S}}$: Intrinsic (relativistic quantum

General angular momentum $\hat{\vec{J}}$

$$\Rightarrow \boxed{\vec{u}_n \cdot \hat{\vec{J}} \equiv \text{Generator of rotations about unit axis } \vec{u}_n}$$

- Infinitesimal symmetry $\hat{D}(\vec{u}_n, \delta\theta) = \hat{\mathbb{1}} - i\delta\theta\,\dfrac{\vec{u}_n \cdot \hat{\vec{J}}}{\hbar}$

- Finite rotation: $\hat{D}(\vec{u}_n, \theta) = e^{-i\theta\,\vec{u}_n \cdot \hat{\vec{J}}/\hbar}$

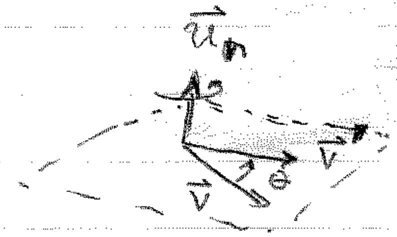
- Rotations

We now add rotations to our list of continuous symmetries. This is extremely important in understanding the three dimensional nature of space, which impacts strongly on the spectrum of energy levels in matter and fields.

Classically, rotations described by orthogonal matrix with unit determinant

In some basis

Rotation about axis $\vec{u}_n$ angle θ (right hand rule)



$$V_i' = R_{ij} V_j$$

$$R_{ij} R^T_{jk} = \delta_{ik}$$

(inner product is preserved)

e.g. rotation about z-axis

$$\mathbb{R} \doteq \begin{array}{cc} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ +\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{matrix} x \\ y \\ z \end{matrix} \end{array}$$

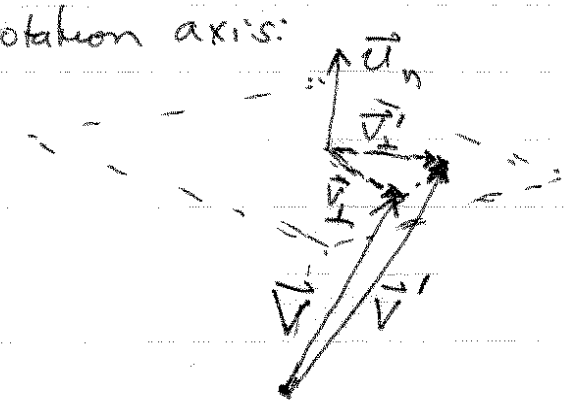
eigenvectors

$\vec{e}_\pm = \mp \dfrac{(\vec{e}_x \pm i\vec{e}_y)}{\sqrt{2}}$

$\vec{e}_0 = \vec{e}_z$

eigenvalues $(e^{\pm i\theta}, 1)$

General rotation axis:



Only component $V_\perp$ is rotated

Vector Operators

Consider a "vector operator" $\hat{\vec{V}}$ (e.g. position $\hat{\vec{X}}$ or momentum $\hat{\vec{p}}$). (Note: there are two different vector spaces to keep straight in your head: 3D physical space and the abstract Hilbert space)

The symmetry operation:

$$\boxed{\hat{D}^{\dagger}(\theta, \vec{u}_n)\ \hat{\vec{V}}\ \hat{D}(\theta, \vec{u}_n) = \hat{\vec{V}}' \quad \text{where} \quad \hat{V}_i' = R_{ij}\ \hat{V}_j \qquad \mathbb{R}\mathbb{R}^T = \mathbb{1}}$$

Infinitesimal version

$$\left(\hat{\mathbb{1}} + \frac{i\delta\theta}{\hbar}\ \vec{u}_n\cdot\hat{\vec{J}}\right)\hat{\vec{V}}\left(\hat{\mathbb{1}} - \frac{i\delta\theta}{\hbar}\ \vec{u}_n\cdot\hat{\vec{J}}\right) = \hat{\vec{V}}'$$

$$= \hat{\vec{V}} + \delta\theta\ \vec{u}_n\times\hat{\vec{V}}$$

Keeping terms on left hand side to first order in $\delta\theta$

$$\Rightarrow \hat{\vec{V}} + i\frac{\delta\theta}{\hbar}\left[\vec{u}_n\cdot\hat{\vec{J}}, \hat{\vec{V}}\right] = \hat{\vec{V}} + \delta\theta\ \vec{u}_n\times\hat{\vec{V}}$$

$$\Rightarrow \left[\vec{u}_n\cdot\hat{\vec{J}}, \hat{\vec{V}}\right] = -i\hbar\ \vec{u}_n\times\hat{\vec{V}}$$

Let $\vec{u}_n = \vec{e}_i$ (Cartesian axis) $\Rightarrow$ $\boxed{[\hat{J}_i, \hat{V}_j] = i\hbar\ \epsilon_{ijk}\hat{V}_k \quad \text{(sum over k)}}$

Angular momentum algebra

Angular momentum itself is a vector under rotations

$$\Rightarrow \boxed{[\hat{J}_i, \hat{J}_j] = i\hbar\,\epsilon_{ijk}\hat{J}_k}$$ Fundamental commutation relations

(e.g. orbital angular momentum, $\hat{L} = \hat{x} \times \hat{p}$ satisfies these)

$\Rightarrow$ Rotations about different axes don't commute.

Consider infinitesimal version:

$$[\hat{D}(\vec{u}_{n_1}, \delta\theta), \hat{D}(\vec{u}_{n_2}, \delta\theta)] = [\hat{\mathbb{1}} - \tfrac{i}{\hbar}\delta\theta\,\hat{\vec{J}}\cdot\vec{u}_{n_1},\ \hat{\mathbb{1}} - i\tfrac{\delta\theta}{\hbar}\,\hat{\vec{J}}\cdot\vec{u}_{n_2}]$$

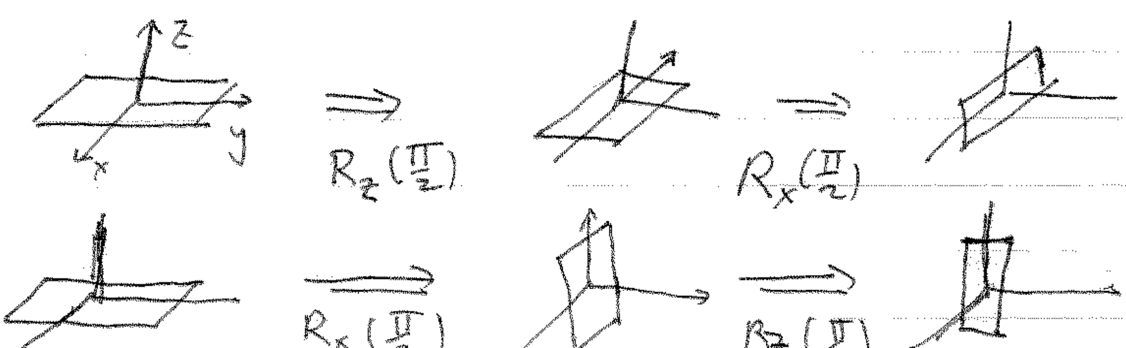
$$= -\frac{(\delta\theta)^2}{\hbar^2}[\hat{\vec{J}}\cdot\vec{u}_{n_1}, \hat{\vec{J}}\cdot\vec{u}_{n_2}] = -\frac{(\delta\theta)^2}{\hbar^2} u^i_{n_1} u^j_{n_2} [\hat{J}_i, \hat{J}_j]$$

$$= -\frac{(\delta\theta)^2}{\hbar^2} u^i_{n_1} u^j_{n_2}\, i\hbar\,\epsilon_{ijk}\hat{J}_k$$

$$\Rightarrow \boxed{[\hat{D}(u_{n_1}, \delta\theta), \hat{D}(u_{n_2}, \delta\theta)] = (\delta\theta)^2 (\vec{u}_{n_1} \times \vec{u}_{n_2}) \cdot \frac{\hat{\vec{J}}}{i\hbar}}$$

Rotations commute for parallel or antiparallel axes only

True classically:



Uncertainty and simultaneous eigenvectors

The Cartesian components $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$ don't commute $\Longrightarrow$ No simultaneous set of eigenvectors

Uncertainty principle:

$$\langle \Delta \hat{J}_x^2 \rangle \langle \Delta \hat{J}_y^2 \rangle \geq \frac{1}{4} \left| \langle [\hat{J}_x, \hat{J}_y] \rangle \right|$$

$$\boxed{\Delta J_x^2 \, \Delta J_y^2 \geq \frac{\hbar}{4} \left| \langle J_z \rangle \right|}$$

Note: RHS is <u>state dependent</u>

However $\hat{J}_i$ commutes with $\hat{\vec{J}}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$

Proof

$$[\hat{J}_i, \hat{\vec{J}}^2] = [\hat{J}_i, \hat{J}_j \hat{J}_j] \quad (\text{Sum over } j)$$

$$= \hat{J}_j [\hat{J}_i, \hat{J}_j] + [\hat{J}_i, \hat{J}_j] \hat{J}_j$$

$$= i\hbar \epsilon_{ijk} \left(\hat{J}_j \hat{J}_k + \hat{J}_k \hat{J}_j \right) = 0$$

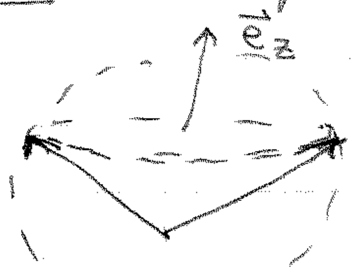
since ϵ_{ijk} anti-sym

$$\Rightarrow \boxed{[\hat{\vec{J}}^2, \hat{J}_i] = 0}$$

$\Rightarrow$ Can simultaneously specify basis of states as eigenstates of $\hat{\vec{J}}^2$ (magnitude of $\vec{J}$ square) and any <u>one</u> component $\hat{J}_i$

<u>Vector picture</u>

$\vec{J}$ with fixed magnitude and one component (here $\vec{e}_z$ is fixed!)



Spherical basis

Since we can only specify eigenstates of one component of $\hat{\vec{J}}$ at a time, the special axis we choose is called the "quantization" axis. Typically this is chosen to be the z-axis.

We saw the for rotations about $\vec{e}_z$, the "spherical basis" $x \pm iy$, z have unique symmetries.
We thus consider

$$\boxed{\hat{J}_{\pm} \equiv \hat{J}_x \pm i\hat{J}_y}$$

"Ladder operators"
$\hat{J}_+^\dagger = \hat{J}_-$

Commutations in the spherical basis $\{\hat{J}_\pm, \hat{J}_z\}$

$$[\hat{J}_z, \hat{J}_\pm] = \underbrace{[\hat{J}_z, \hat{J}_x]}_{+i\hbar\hat{J}_y} \pm i\underbrace{[\hat{J}_z, \hat{J}_y]}_{-i\hbar J_x} = \pm\hbar(\hat{J}_x \pm i\hat{J}_y)$$

$$\Rightarrow \quad \boxed{[\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm}$$

$$[\hat{J}_+, \hat{J}_-] = [\hat{J}_x + i\hat{J}_y, \hat{J}_x - i\hat{J}_y] = -i[\hat{J}_x, \hat{J}_y] + i[\hat{J}_y, \hat{J}_x]$$

$$= -2i[\hat{J}_x, \hat{J}_y] = -2i(i\hbar\hat{J}_z) =$$

$$\Rightarrow \boxed{[\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z}$$

and $\boxed{[\hat{\vec{J}}^2, \hat{J}_z] = [\hat{\vec{J}}^2, \hat{J}_\pm] = 0}$