Lecture 6 — Angular Momentum Eigenvalue Problem

Dimensionless operators

To simplify the algebra, we define dimensionless versions of the angular momentum operators:

$$\text{let } \hat{\vec{j}} \equiv \frac{\hat{\vec{J}}}{\hbar}$$

$$\Rightarrow \boxed{\begin{array}{ll} [\hat{j}_i, \hat{j}_j] = i\epsilon_{ijk}\,\hat{j}_k & [\hat{\vec{j}}^{\,2}, \hat{j}_i] = 0 \\ [\hat{j}_+, \hat{j}_-] = 2\hat{j}_z & [\hat{j}_z, \hat{j}_\pm] = \pm\hat{j}_\pm \\ \multicolumn{2}{c}{[\hat{j}^2, \hat{j}_\pm] = 0} \end{array}}$$

We seek the eigenvalues and eigenvectors of these operators. Since $[\hat{j}_i, \hat{j}_j] \neq 0$ we cannot simultaneously diagonalize any two components of $\hat{\vec{j}}$. However since $[\hat{\vec{j}}^{\,2}, \hat{j}_i] = 0$ we can find ~~sim~~ a common basis of eigenvectors of $\hat{\vec{j}}^{\,2}$ and any one component $\hat{j}_i$. We typical call this component $\hat{j}_z$, with "z" the "quantization axis"

(Next Page)

Eigenvalue problem

We thus seek solutions:

$$\hat{J}^2 |\lambda, m\rangle = \lambda \; |\lambda, m\rangle$$

$$\hat{J}_z \underbrace{|\lambda, m\rangle}_{\text{state labeled by two eigenvalues}} = m |\lambda, m\rangle$$

In order to procede we need a few more relations

Note: $\hat{J}_+\hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) = \hat{J}_x^2 + \hat{J}_y^2 + i[\hat{J}_y, \hat{J}_x]$

but $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$ and $[\hat{J}_y, \hat{J}_x] = -i\hat{J}_z$

$$\Rightarrow \boxed{\hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_z^2 + \hat{J}_z = \hat{J}^2 - \hat{J}_z(\hat{J}_z - 1)}$$

$$\boxed{\text{Similarly } \hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z(\hat{J}_z + 1)}$$

Lemma: $\lambda \geq m^2$

Proof: $\langle \lambda m | \hat{J}^2 | \lambda m \rangle = \lambda = \langle \lambda m | (\hat{J}_x^2 + \hat{J}_y^2) | \lambda m \rangle$
$+ \langle \lambda m | \hat{J}_z^2 | \lambda m \rangle$

$$\Rightarrow \lambda = \underbrace{\langle \lambda m | \hat{J}_x^2 + \hat{J}_y^2 | \lambda m \rangle}_{\geq 0} + m^2$$

q.e.d

$\Rightarrow \exists$ a maximum m_{max} and minimum m_{min} for a given λ

Lemma: $\hat{J}_+$ and $\hat{J}_-$ are raising & lowering operators for m, respectively

i.e. $\hat{J}_z(\hat{J}_+|\lambda m\rangle) = m+1\ (\hat{J}_+|\lambda m\rangle)$

$\hat{J}_z(\hat{J}_-|\lambda m\rangle) = m-1\ (\hat{J}_-|\lambda m\rangle)$

$\Rightarrow \hat{J}_+|\lambda m\rangle \propto |\lambda, m+1\rangle \qquad \hat{J}_-|\lambda m\rangle \propto |\lambda, m-1\rangle$

Proof: $\hat{J}_z\hat{J}_\pm|\lambda m\rangle = (\hat{J}_\pm\hat{J}_z + [\hat{J}_z, \hat{J}_\pm])\,|\lambda m\rangle$

$= \hat{J}_\pm\underbrace{\hat{J}_z|\lambda m\rangle}_{m|\lambda m\rangle} \pm \hat{J}_\pm|\lambda m\rangle = (m\pm 1)\ (\hat{J}_\pm|\lambda m\rangle)$

q.e.d.

Consider now the maximum value of $m \equiv m_{max}$
$\Rightarrow \hat{J}_+|\lambda, m_{ax}\rangle = 0$

$\hat{J}^2|\lambda, m_{max}\rangle = \lambda\,|\lambda, m_{max}\rangle$

$\Downarrow$

$= (\hat{J}_-\hat{J}_+ + \hat{J}_z(\hat{J}_z+1))\,|\lambda, m_{ax}\rangle = m_{max}(m_{max}+1)\ |\lambda, m_{max}\rangle$

$\Rightarrow \boxed{\lambda = m_{max}(m_{max}+1)}$

Consider minimum $m \Rightarrow \hat{J}_-|\lambda, m_{min}\rangle = 0$

$\hat{J}^2|\lambda, m_{min}\rangle = \lambda|\lambda, m_{min}\rangle = (\hat{J}_+\hat{J}_- + \hat{J}_z(\hat{J}_z-1))\,|\lambda, m_{min}\rangle$

$= m_{min}(m_{min}-1)|\lambda, m_{min}\rangle \Rightarrow \boxed{\lambda = m_{min}(m_{min}-1)}$

Thus: $m_{max}(m_{max}+1) = m_{min}(m_{min}-1)$

$\Rightarrow \quad m_{max} = -m_{min} \equiv j \leftarrow$ definition

$\therefore$ Solution $\lambda = j(j+1)$

$-j \leq m \leq j$ in integer steps

$\Rightarrow 2j+1$ m-values $=$ integer

$\Rightarrow j =$ integer of $\frac{1}{2}$-integer

eigenvector: $|j, m\rangle$

$j =$ integer: orbital angular momentum

$j = \frac{1}{2}$ integer: spin angular momentum

Raising and lowering operators

$\hat{J}_+|j\,m\rangle = c_{m,+}\,|j, m+1\rangle$

$\|\hat{J}_+$ $\quad \langle j m|\hat{J}_-\hat{J}_+|j,m\rangle = |c_{m,+}|^2$

$= \langle j, m|\hat{J}^2 - \hat{J}_z(\hat{J}_z+1)|j, m\rangle = j(j+1) - m(m+1)$

Phase convention $\Rightarrow$ $c_{m,+} = \sqrt{j(j+1) - m(m+1)}$

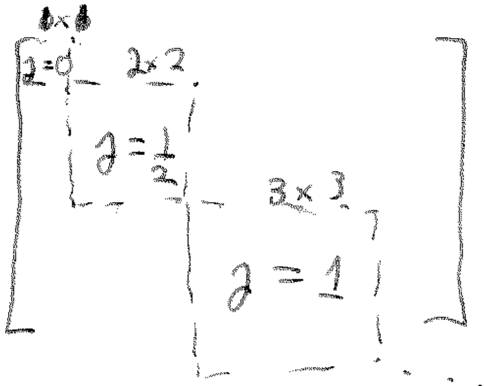
$\Rightarrow$ $\hat{J}_+|j, m\rangle = \sqrt{j(j+1) - m(m+1)}\;|j, m\rangle$

Similarly $\hat{J}_-|j, m\rangle = \sqrt{j(j+1) - m(m-1)}\;|j, m\rangle$

Representation in the standard basis:

$$\{ |j,m\rangle \mid j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ \ldots \ , -j \le m \le j \}$$

$\Rightarrow (2j+1) \times (2j+1)$ blocks



No off-diagonal elements between different j's

Matrix elements

$$\boxed{\begin{aligned}
&\langle j, m' | \hat{J}_z | j, m\rangle = m\, \delta_{m m'} \qquad \text{diagonal} \\
&\langle j, m' | \hat{J}_x | j, m\rangle = \tfrac{1}{2}\left(\langle j, m' | \hat{J}_+ + \hat{J}_- | j\, m\rangle \right) \\
&\quad = \tfrac{1}{2}\left(\sqrt{j(j+1) - m(m+1)}\, \delta_{m', m+1} + \sqrt{j(j+1) - m(m-1)}\, \delta_{m', m-1} \right) \\
&\langle j, m' | \hat{J}_y | j, m\rangle = \\
&\quad \frac{-i}{2}\left(\sqrt{j(j+1) - m(m+1)}\, \delta_{m', m+1} - \sqrt{j(j+1) - m(m-1)}\, \delta_{m', m-1} \right)
\end{aligned}}$$

Examples: (restrict to $(2j+1)\times(2j+1)$ subspace)

- $j = 1/2 \qquad m_j = -1/2,\ 1/2$

$$\begin{array}{cl} \begin{array}{cc} m=1/2 & m=1/2 \end{array} & \\ \hat{J}_z \doteq \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix} & \begin{array}{l} m'=1/2 \\ m'=-1/2 \end{array} \end{array}$$

(ordered basis)

$$\hat{J}_x \doteq \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \qquad \hat{J}_y \doteq \begin{bmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{bmatrix}$$

note $\hat{J}_i = \frac{1}{2}\hat{\sigma}_i$: Pauli matrices

- $j = 1, \qquad m_j = -1, 0, 1$

$$\begin{array}{cl} \begin{array}{ccc} m=1 & 0 & -1 \end{array} & m' \\ \hat{J}_z \doteq \begin{bmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} & \begin{array}{l} 1 \\ 0 \\ -1 \end{array} \end{array}$$

$$\hat{J}_x \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \hat{J}_y \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

Note: $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \doteq 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$