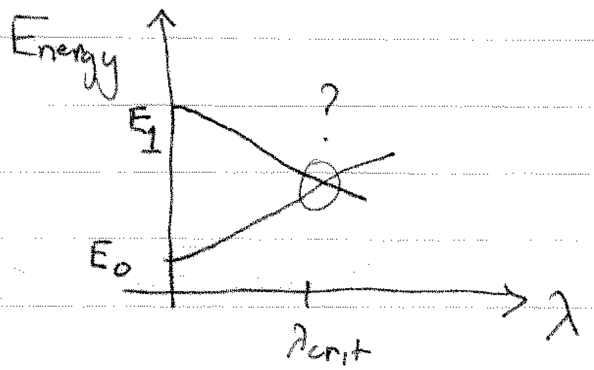
Lecture 14

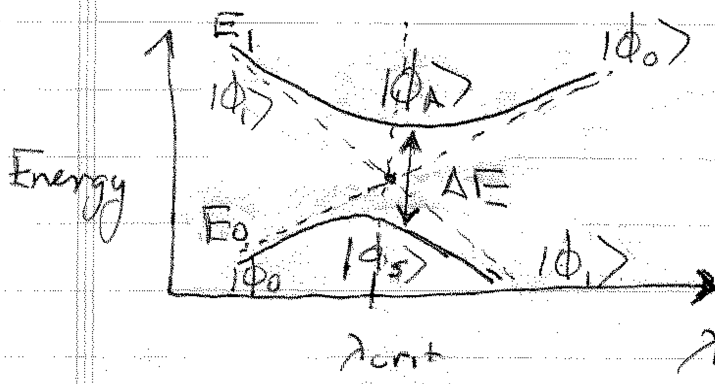
"Anti-crossing" : Behavior of coupled 2 level system

Suppose we perturb a system by $\hat{H}_{int}(\lambda)$ where λ is some parameter (e.g. field strength)



We may run into a situation where two previously non-degenerate states are now made near degenerate. Do they actually cross?

If state $|\phi_0\rangle$ and $|\phi_1\rangle$ "talk to one another", i.e. there is a coupling matrix element between them, there is no degeneracy $\Rightarrow$ anti crossing



Close up

Generic behavior at an anti-crossing.

The splitting $\Delta E \propto \langle\phi_1|\hat{H}_{int}|\phi_0\rangle$

Note: Through an anti-crossing, the nature of the state changes

$$|\phi_{A \atop S}\rangle = \frac{|\phi_1\rangle \mp |\phi_0\rangle}{\sqrt{2}}$$: Superposition at the crossing

Generic behavior at an anti-crossing: Spin 1/2 particle in a magnetic field (exactly solvable)

Consider magnetic resonance problem:

- Variable $\vec{B}$ field in z-direction.
- Weak transverse field in x-y plane.

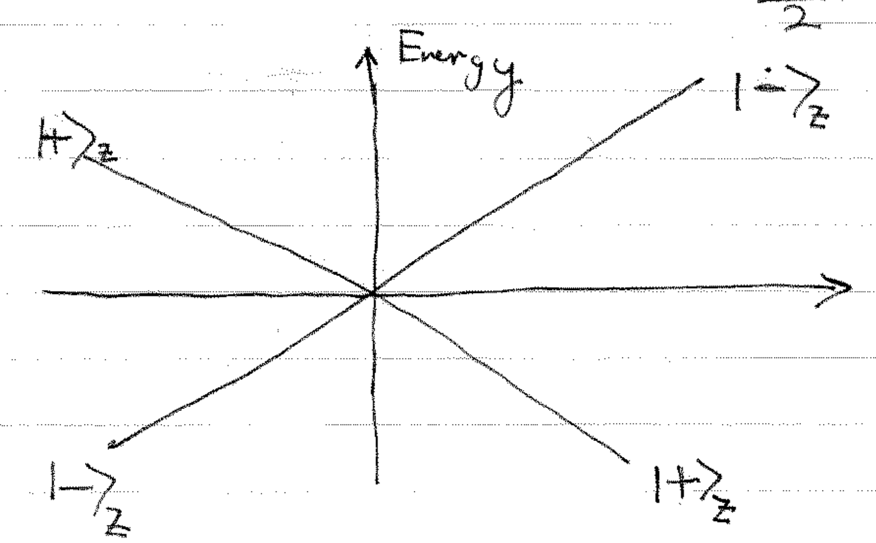
$$\vec{B}_{total} = \vec{B}_{\parallel} + \vec{B}_{\perp} \qquad \vec{B}_{\parallel} = B_z \vec{e}_z$$
$$\vec{B}_{\perp} = B_x \vec{e}_x + B_y \vec{e}_y$$

Hamiltonian: $\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B} = -\frac{\hbar\gamma}{2} \hat{\vec{\sigma}} \cdot \vec{B}$ (γ = gyromagnetic ratio)

$$= -\frac{\hbar \vec{\Omega}}{2} \cdot \hat{\vec{\sigma}} \qquad \vec{\Omega} = \gamma \vec{B}$$

Larmor / Rabi frequency

Suppose $\vec{B}_{\perp} = 0 \Rightarrow \hat{H} = -\frac{\hbar\Omega}{2} \hat{\sigma}_z$



Eigenvalues
$\mp \frac{\hbar \Omega_{\parallel}}{2}$

Eigenvectors
$|+\rangle_z$ and $|-\rangle_z$

Energy splitting $\Delta E = \hbar\Omega$

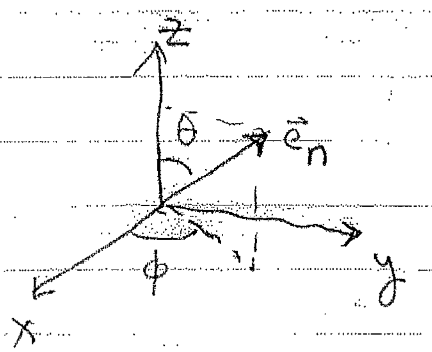
Now turn on $\vec{B}_\perp$ $\Rightarrow$ $\vec{\Omega} = \Omega_{\parallel}\vec{e}_z + \vec{\Omega}_\perp \vec{e}_\perp$

$$= \Omega_x \vec{e}_x + \Omega_y \vec{e}_y + \Omega_{\parallel}\vec{e}_z$$

$$\hat{H} = -\frac{\hbar \vec{\Omega}}{2}\cdot\hat{\vec{\sigma}} = -\frac{\hbar\tilde{\Omega}}{2}\hat{\sigma}_n$$

where $\vec{\Omega} = \tilde{\Omega}\,\vec{e}_n$ $\quad$ $\tilde{\Omega} = |\vec{\Omega}| = \sqrt{\Omega_{\parallel}^2 + |\vec{\Omega}_\perp|^2}$

$$\vec{e}_n = \frac{\vec{\Omega}}{\tilde{\Omega}} = \frac{\Omega_{\parallel}}{\tilde{\Omega}}\vec{e}_z + \frac{\Omega_x}{\tilde{\Omega}}\vec{e}_x + \frac{\Omega_y}{\tilde{\Omega}}\vec{e}_y$$



$$\vec{e}_n = \cos\theta\,\vec{e}_z + \sin\theta\left(\cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y\right)$$

$$\tan\theta = \frac{\Omega_\perp}{\Omega_{\parallel}} \qquad \text{"}\theta = \text{mixing angle"}$$

$$\tan\phi = \frac{\Omega_x}{\Omega_y} \qquad \phi = \text{phase}$$

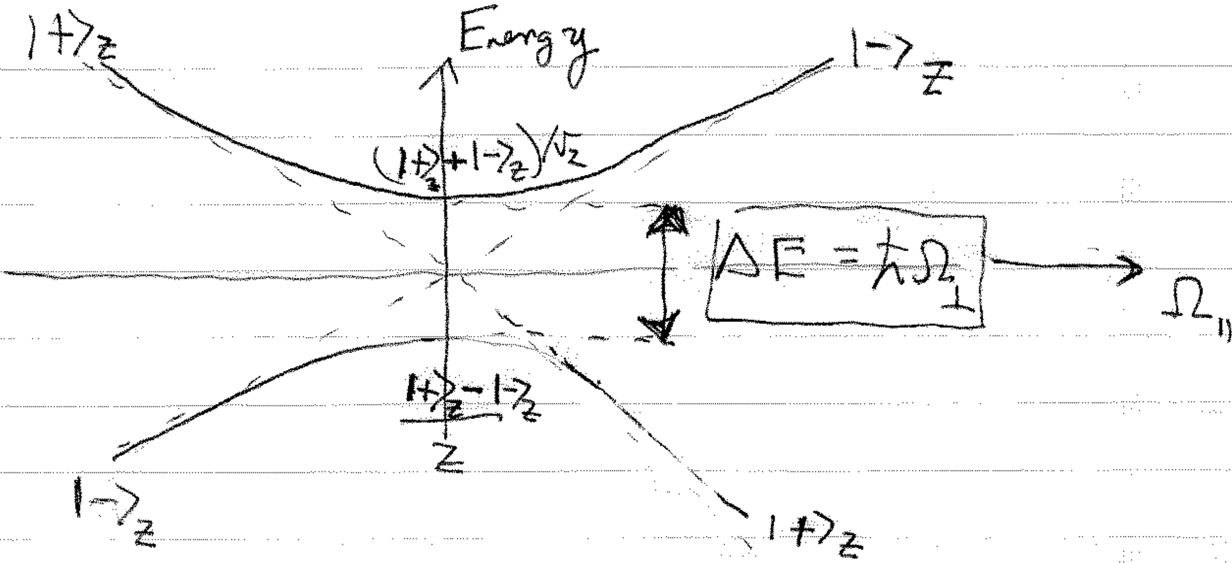
Eigenstates: given phase convention
$$\begin{cases} |+\rangle_n = \cos\frac{\theta}{2}|+\rangle_z + e^{i\phi}\sin\frac{\theta}{2}|-\rangle_z \\ |-\rangle_n = \sin\frac{\theta}{2}|+\rangle_z - e^{i\phi}\cos\frac{\theta}{2}|-\rangle_z \end{cases}$$

Eigenvalue $\hat{\sigma}_n$: ± 1

$\Rightarrow$ Eigenvalues of $\hat{H} = \mp\frac{\hbar\tilde{\Omega}}{2}$

With $\Omega_\perp$ fixed the equation for the energy eigenvalues is a hyperbola

$$E_\pm = \mp\frac{\hbar}{2}\sqrt{\Omega_{\parallel}^2 + \Omega_\perp^2}$$



Far from degeneracy point $\frac{\hbar\Omega_\perp}{2}\hat{\sigma}_\perp$ is a perturbation

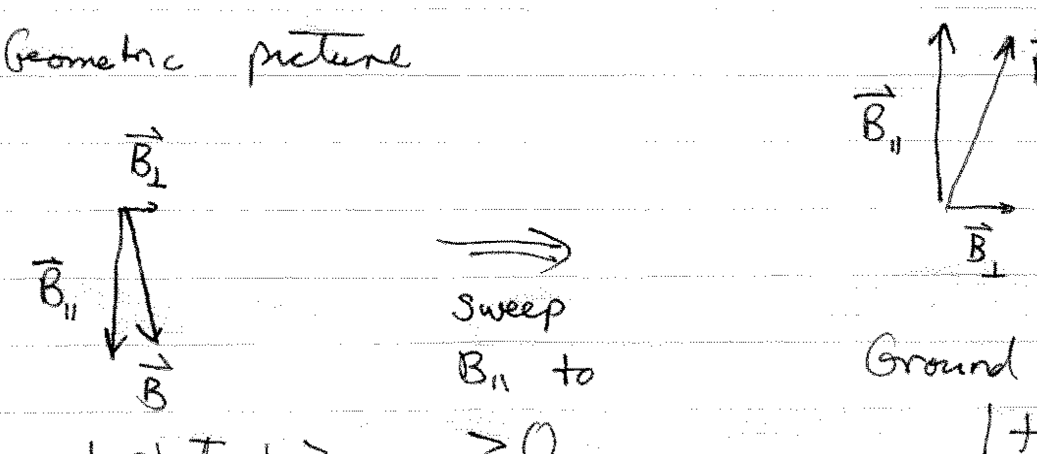
$$\Rightarrow \quad E_\pm \approx \mp \frac{\hbar\Omega_\parallel}{2} \mp \frac{\hbar}{4}\frac{\Omega_\perp^2}{\Omega_\parallel}$$

At degeneracy we have an anti-crossing

$\Rightarrow$ States mix to equal superpositions of bare states

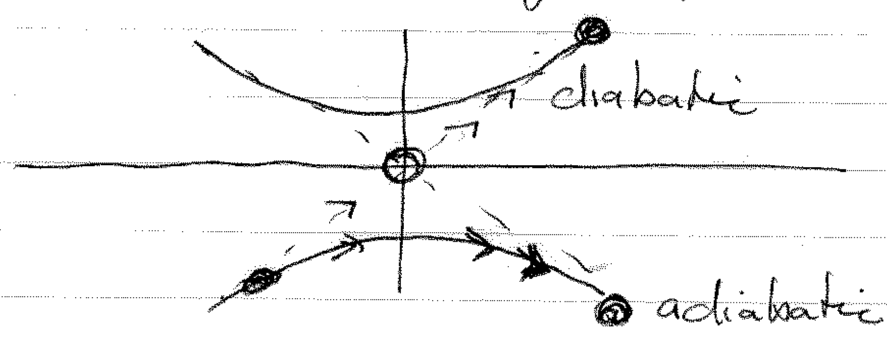
- <u>Note</u>: The nature of the eigenstate flips through the anti crossing

Geometric picture

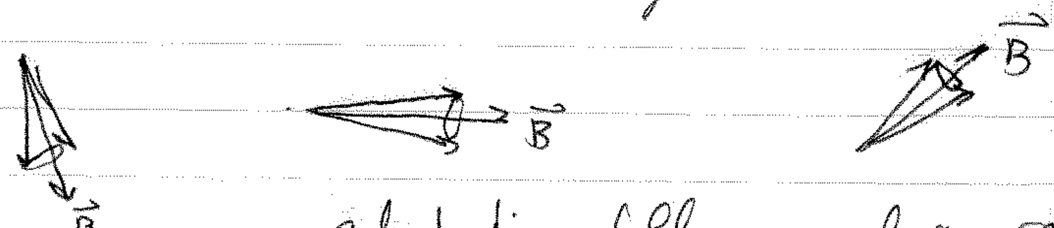


Ground state $|-\rangle_z$ Ground state $|+\rangle_z$

Adiabatic theorem: If parameter changes much more slowly than (energy spacing)/$\hbar$ $\Rightarrow$ adiabatic following of energy level



If the change is ~~too~~ quick we can apply the "sudden" approximation in which case the state does not change $\Rightarrow$ "diabatic" transition



adiabatic following of a spin

Generic two-dimension Hilbert-Space $\Rightarrow$ pseudo-spin

$$\hat{H} \doteq \begin{bmatrix} a & \epsilon \\ \epsilon^* & b \end{bmatrix} = \frac{A}{2}\hat{\mathbb{1}} + \vec{B}\cdot\frac{\vec{\sigma}}{2}$$

$$A = \mathrm{Tr}(\hat{H}) = a+b$$

$$\vec{B} = \mathrm{Tr}(\hat{H}\vec{\sigma}) \Rightarrow \quad B_x = 2\,\mathrm{Re}\,\epsilon \quad B_y = 2\,\mathrm{Im}\,\epsilon$$
$$B_z = a-b$$

Mixing angle $\dfrac{2|\epsilon|}{a-b} = \tan\theta \qquad \tan\phi = \dfrac{\mathrm{Im}\,\epsilon}{\mathrm{Re}\,\epsilon}$