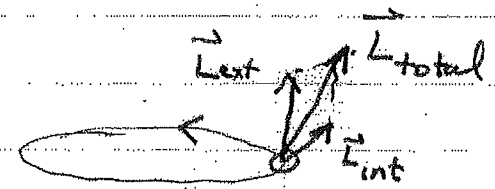
Lecture 15: Addition of Angular Momentum

Classically, angular momentum adds as a vector



e.g. Spinning top with internal angular momentum $\vec{L}_{int}$ in orbit with $\vec{L}_{ext}$

$$\vec{L}_{total} = \vec{L}_{int} + \vec{L}_{ext}$$

Quantum mechanically, the situation is more complex since different components of angular momentum do not commute

Generally problem:

Given two angular momenta $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ (e.g. spin + orbital of a given particle, two spins, etc.).

$\hat{\vec{J}}_1$ acts on h_1, spanned by $|j_1, m_1\rangle$ $(2j_1+1)$-dim

$\hat{\vec{J}}_2$ acts on h_2, spanned by $|j_2, m_2\rangle$ $(2j_2+1)$-dim

Composite space $\mathcal{H} = h_1 \otimes h_2$, spanned by the basis $|j_1 m_1\rangle \otimes |j_2 m_2\rangle = |j_1 m_1 ; j_2 m_2\rangle$

(total of $(2j_1+1)(2j_2+1)$ basis vectors).

The basis vectors are simultaneous eigenstates of

$$\{\hat{J}_1^2, \hat{J}_{1z}, \hat{J}_2^2, \hat{J}_{2z}\} = \text{Complete set of mutually commuting Operators}$$

"Uncoupled representation"

Consider the "total angular momentum" operator

$$\hat{\vec{J}} \equiv \hat{\vec{J}}_1 + \hat{\vec{J}}_2 \quad \text{on} \quad \mathcal{H}$$

(really $\hat{\vec{J}} = \hat{\vec{J}}_1 \otimes \hat{\mathbb{1}}_2 + \hat{\mathbb{1}}_1 \otimes \hat{\vec{J}}_2$)

components $\hat{J}_i = \hat{J}_{1_i} + \hat{J}_{2_i}$

$$\hat{J}^2 = |\hat{\vec{J}}_1 + \hat{\vec{J}}_2|^2 = \hat{J}_1^2 + \hat{J}_2^2 + 2\hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2$$

Does $\hat{\vec{J}}$ satisfy angular momentum algebra?

$$[\hat{J}_i, \hat{J}_j] = [\hat{J}_{1_i}, \hat{J}_{1_j}] + [\hat{J}_{2_i}, \hat{J}_{2_j}]$$

(since $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ commute)

$$= i\epsilon_{ijk}\hat{J}_{1_k} + i\epsilon_{ijk}\hat{J}_{2_k} = i\epsilon_{ijk}(\hat{J}_{1_k} + \hat{J}_{2_k})$$

$$\Rightarrow \boxed{[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k} \quad \checkmark$$

$$[\hat{J}^2, \hat{J}_z] = [\hat{J}_1^2 + \hat{J}_2^2 + 2\hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2, \ \hat{J}_{1z} + \hat{J}_{2z}]$$

$$= \underbrace{[\hat{J}_1^2, \hat{J}_{1z}]}_{= 0} + \underbrace{[\hat{J}_2^2, \hat{J}_{2z}]}_{= 0} + 2\left([\hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2, \hat{J}_{1z}] + [\hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2, \hat{J}_{2z}]\right)$$

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Aside: $[\hat{\vec{J}}_1\cdot\hat{\vec{J}}_2, \hat{J}_{1z}] = [\hat{\vec{J}}_2\cdot\hat{\vec{J}}_1, \vec{e}_z\cdot\hat{\vec{J}}_1]$

$$= i\,(\hat{\vec{J}}_2\times\vec{e}_z)\cdot\hat{\vec{J}}_1$$

$$[\hat{\vec{J}}_1\cdot\hat{\vec{J}}_2, \hat{J}_{2z}] = [\hat{\vec{J}}_1\cdot\hat{\vec{J}}_2, \vec{e}_z\cdot\hat{\vec{J}}_2]$$

$$= i\,(\hat{\vec{J}}_1\times\vec{e}_z)\cdot\hat{\vec{J}}_2 = -i\,(\hat{\vec{J}}_2\times\vec{e}_z)\cdot\hat{\vec{J}}_1$$

$$\therefore \quad \boxed{[\hat{J}^2, \hat{J}_z] = 0}$$

Thus $\hat{\vec{J}}$ is an angular moment operator

Note $[\hat{J}^2_{\frac{1}{2}}, \hat{J}_z] = [\hat{J}^2_{\frac{1}{2}}, \hat{J}_{1z}+\hat{J}_{2z}] = [\hat{J}^2_{\frac{1}{2}}, \hat{J}_{\frac{1z}{2}}] = 0$

$\therefore \exists$ simultaneous eigenstates of the total angular momentum operators $\{\hat{J}^2, \hat{J}_z\}$

$$\{|j,m\rangle\}$$

These are also eigenstates of $\hat{J}_1^2$ and $\hat{J}_2^2$

Complete set of commuting operators

$$\{\hat{J}^2, \hat{J}_1^2, \hat{J}_2^2, \hat{J}_z\}$$

$\Rightarrow$ Eigenstates $\{|j,m; j_1, j_2\rangle\}$

"Coupled representation"

<u>However</u>: $[\hat{j}^2, \hat{j}_{1z}] = i(\hat{\vec{j}}_2 \times \vec{e}_z)\cdot \hat{\vec{j}}_1 \neq 0$

$$[\hat{j}^2, \hat{j}_{2z}] = i(\hat{\vec{j}}_1 \times \vec{e}_z)\cdot \hat{\vec{j}}_2 \neq 0$$

$\Rightarrow$ state $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$ is generally <u>not</u> eigenstate of $\hat{j}^2$

$\Rightarrow$ <u>Two different representations</u>

- <u>Uncoupled representation</u>: $\{|j_1 m_1; j_2 m_2\rangle\}$

  ~~Simulatane~~ Simultaneous eigenkets of $\{\hat{j}_1^2, \hat{j}_{1z}, \hat{j}_2^2, \hat{j}_{2z}\}$

- <u>Coupled representation</u>: $\{|j m; j_1, j_2\rangle\}$

  Simultaneous eigenkets of $\{\hat{j}^2, \hat{j}_z, \hat{j}_1^2, \hat{j}_2^2\}$

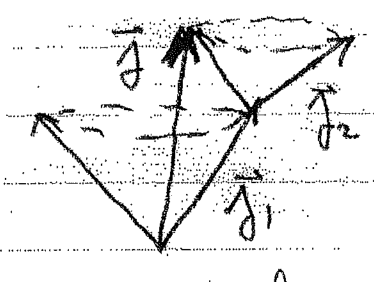
Note $\hat{j}^2 = \hat{j}_1^2 + \hat{j}_2^2 - 2\hat{j}_1\cdot\hat{j}_2$

$\Rightarrow |j_1 - j_2| \leq j \leq j_1 + j_2$ triangle inequality

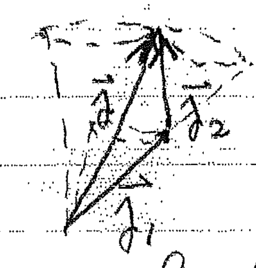
(steps of 1)

Total # of states: $\sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) = (2j_1+1)(2j_2+1)$ ✓

<u>Vector picture</u>



Uncoupled



Coupled

Example: Two spins ($s = 1/2$)

- Uncoupled: $\{ \hat{S}_1^2, \hat{S}_{1z} ; \hat{S}_2^2, \hat{S}_{2z} \}$

Basis $\{ |+_z\rangle \otimes |+_z\rangle, |+_z\rangle \otimes |-_z\rangle, |-_z\rangle \otimes |+_z\rangle, |-_z +_z\rangle \}$

$= \{ |++\rangle, |+-\rangle, |-+\rangle, |--\rangle \}$ 4-dim

- Coupled: $\{ \hat{S}^2, \hat{S}_z, \hat{S}_1^2, \hat{S}_2^2 \}$ $\quad \hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$

Basis $\{ |s, m; s_1 s_2\rangle \}$

$|s_1 - s_1| \leq s \leq s_1 + s_2 \Rightarrow 0 < s < 1$

- $s = 0 \Rightarrow |s=0, m=0\rangle$ "singlet"
- $s = 1 \Rightarrow |s=1, m=-1\rangle, |s=1, m=0\rangle, |s=1, m=1\rangle$

"triplet"

(Note since s_1 and s_2 are fixed, we need not write them)

Since either basis is a complete set we can expand one set in terms of the other

$$|s, m\rangle = \sum_{\substack{m_1, m_2 \\ = \pm 1/2}} C^{sm}_{s_1 m_1, s_2 m_2} |s_1 m_1\rangle \otimes |s_2 m_2\rangle$$

Clebsch-Gordan coefficient

Find coefficients which diagonize $\hat{S}^2$ and $\hat{S}_z$

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$\Rightarrow$ Diagonalize the matrix representation of $\hat{S}^2$ and $\hat{S}_z$ in $\{|s_1 m_1\rangle \otimes |s_2 m_2\rangle\}$

Note: $\hat{S}_z (|s_1 m_1\rangle \otimes |s_2 m_2\rangle) = (\hat{S}_{1z} + \hat{S}_{2z})(|s_1 m_1\rangle \otimes |s_2 m_2\rangle)$

$$= (m_1 + m_2)\,(|s_1, m_1\rangle \otimes |s_2\, m_2\rangle)$$

$\Rightarrow$ $\hat{S}_z$ is diagonal in this basis

$$\hat{S}_z \doteq \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} \\ \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{bmatrix}$$

However:

$$\hat{S}^2(|s_1 m_1\rangle \otimes |s_2 m_2\rangle) = (\hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_1 \cdot \hat{S}_2)\,|m_1, m_2\rangle$$

$$= \left(\frac{3}{4} + \frac{3}{4}\right)|m_1, m_2\rangle + 2\hat{S}_1 \cdot \hat{S}_2 |m_1, m_2\rangle$$

$$= \frac{3}{2}|m_1, m_2\rangle + (\hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+} + 2\hat{S}_{1z}\hat{S}_{2z})|m_1, m_2\rangle$$

$$= \left(\frac{3}{2} + 2m_1 m_2\right)|m_1\, m_2\rangle + \hat{S}_{1+}|m_1\rangle \otimes |\hat{S}_{2-}|m_2\rangle$$

$$\hat{S}_{1-}|s_1, m_1\rangle \otimes \hat{S}_{2+}|s_2, m_2\rangle$$

$\therefore$ $\hat{S}^2|++\rangle = 2|++\rangle$ $\qquad$ $\hat{S}^2|+-\rangle = |+-\rangle + |-+\rangle$

$\hat{S}^2|--\rangle = 2|--\rangle$ $\qquad$ $\hat{S}^2|-+\rangle = |-+\rangle + |+-\rangle$

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$$\Rightarrow \hat{S}^2 = \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \\ \end{array}$$
$$\hat{S}^2 = \left[\begin{array}{c|cc|c} 2 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right]$$

Block diagonal: subspace $m=0$:
$$\begin{array}{ccc} & |+-\rangle & |-+\rangle \\ \hat{S}^2 = & \left[\begin{array}{c} 1 \\ 1 \end{array}\right. & \left.\begin{array}{c} 1 \\ 1 \end{array}\right] \end{array}$$

Secular equation:

$$\det(\hat{S}^2 - \lambda \hat{\mathbb{1}}) = \lambda^2 - 2\lambda = 0$$

eigenvalues $\lambda = 0$: $\hat{S}|\lambda\rangle = 0 \Rightarrow s = 0$

$\lambda = 2$: $\hat{S}|\lambda\rangle = 2|\lambda\rangle = s(s+1) \Rightarrow s = 1$

eigenvectors: $s=0$: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \doteq \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$

$s = 1$: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \doteq \frac{1}{\sqrt{2}}(|+-\rangle + |+-\rangle)$

We thus arrive at the relationship between the coupled and uncoupled representations:

$$\boxed{\begin{array}{l} \underline{\text{Singlet:}} \\ |s=0, m=0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \\ \\ \underline{\text{Triplet}} \\ |s=1, m=1\rangle = |++\rangle \\ |s=1, m=0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \\ |s=1, m=-1\rangle = |--\rangle \end{array}}$$

Note: the $|s, m=0\rangle$ are entangled!