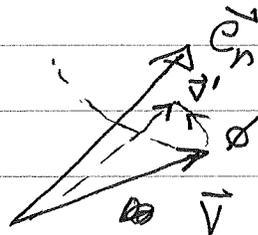# Lecture 18 : The Rotation Group

An important symmetry in understanding the structure
of matter is the rotation group. Let us
begin with a review of the classical theory.

Classical Theory

Rotation in 3D

Axis-angle parametrization

$(\hat{n}, \phi)$ = three parameters : $\begin{cases} 2 \text{ for specifying } \hat{n} \\ \phi \text{ for angle} \end{cases}$

Under a rotation, the coordinates transform as

$$V_i' = R_{ij} \, V_j \qquad \text{(repeated index}$$

$$\uparrow$$
$$\text{rotation matrix} \qquad\qquad \text{sum convention)}$$

The norm of the vector is preserved $\|\vec{V}'\|^2 = \|\vec{V}\|^2$

$$\Rightarrow \quad V_i' V_i' = V_j V_j = (R_{ij} V_j)(R_{ik} V_k)$$

$$\Rightarrow R_{ij} R_{ik} = \delta_{ik} \quad \Rightarrow \boxed{R^T R = \mathbb{1}}$$

Thus rotation matrices are orthonogonal
matrices, i.e. $R^{-1} = R^T$

More over, a rotation operation preserves the "handedness" of the 3D coordinate system. This translates into $\boxed{\det R = 1}$

Such $\det = 1$ matrices are said to be "special".

Thus, the group of rotations on 3D vectors is defined by the set of matrices, 3×3, which are orthogonal and special

$$\boxed{3D \text{ rotation group } = SO(3)}$$

The group $SO(3)$ is a Lie group on a 3D manifold. To understand the Lie algebra, consider an infinitesimal rotation, $\delta\phi$

$$R(\vec{n}, \delta\phi) = \hat{1} + \delta\phi \, \hat{A}(\vec{n})$$
$\underset{\sim}{\text{infinitesimal generator}}$

$$R^T(\vec{n}, \delta\phi) \, R(\vec{n}, \delta\phi) = \hat{1} \implies \hat{1} + \delta\phi(\hat{A} + \hat{A}^T) = \hat{1}$$

$$\implies \boxed{\hat{A} = -\hat{A}^T} \quad \implies \begin{array}{l} \text{The Lie algebra } so(3) \\ \text{is the set of anti-symmetric} \\ \text{matrices, } 3 \times 3 \end{array}$$

A basis for $so(3)$ is $\{\hat{I}^{(1)}, \hat{I}^{(2)}, \hat{I}^{(3)}\}$

$$\hat{I}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \hat{I}^{(2)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \hat{I}^{(3)} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The Lie algebra basis satisfies the commutators

$$\left[\hat{\mathbb{I}}^{(i)}, \hat{\mathbb{I}}^{(j)} \right] = \epsilon_{ijk} \hat{\mathbb{I}}^{(k)}$$

Note: We can also define the algebra with the set of Hermitian matrices with

$$\hat{\mathbb{J}}^{(k)} \equiv -i \hat{\mathbb{I}}^{(k)}$$

Then $$\left[\hat{\mathbb{J}}^{(i)}, \hat{\mathbb{J}}^{(j)} \right] = i \epsilon_{ijk} \hat{\mathbb{J}}^{(k)}$$

In physics, we tend to use the Hermitian matrices because of there connection to quantum mechanics and observables.

In the axis angle parameterization

$$\mathbb{R}(\vec{n}, \delta\phi) = \hat{\mathbb{1}} + \delta\phi \, \vec{n} \cdot \hat{\vec{\mathbb{I}}}$$

The matrix element of $\hat{\mathbb{I}}_{ij}^{(k)} = -\epsilon_{ijk}$

$$\Rightarrow \mathbb{R}(\vec{n}, \phi)_{ij} V_j = \left(\delta_{ij} + \delta\phi \, \mathbb{I}_{ij}^{(k)} n_k \right) V_j$$

$$= \left(\delta_{ij} - \delta\phi \, n_k \, \epsilon_{ijk} \right) V_j$$

$$\Rightarrow \mathbb{R}_{\delta\phi} \vec{V} = \vec{V} + \delta\phi (\vec{n} \times \vec{V}) \quad \text{as expected}$$

For finite rotation $\boxed{\mathbb{R}(\phi, \vec{n}) = e^{\phi \vec{n} \cdot \hat{\vec{\mathbb{I}}}}}$

Quantum Mechanics

Angular momentum operator is the generator of rotation

Infinitesimal: $\hat{U}(\delta\phi) = \hat{1} - i\,\delta\phi\,\vec{e}_n \cdot \hat{\vec{J}}$ $(\hbar = 1)$

$$\Rightarrow \hat{U}^\dagger(\delta\phi)\,\hat{\vec{V}}\,\hat{U}(\delta\phi) = \vec{V} + \delta\phi\,(\vec{e}_n \times \vec{V})$$

$$[\hat{J}_i, \hat{V}_j] = i\epsilon_{ijk}\hat{V}_k$$

Finite rotation
$$\hat{U}(\phi, \vec{e}_n) = e^{-i\phi\,\vec{e}_n \cdot \hat{\vec{J}}}$$

Lie group, Lie algebra of generators
$$[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k \qquad [\hat{J}^2, \hat{J}_i] = 0$$

Also define spherical basis: $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$, $\hat{J}_z$

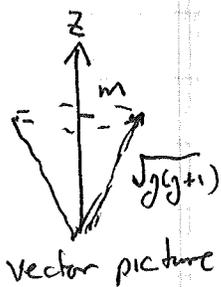$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z \qquad [\hat{J}_z, \hat{J}_\pm] = \pm\hat{J}_\pm$$

Eigenvalue problem: Simultaneous commuting pair
$$\{\hat{J}^2, \hat{J}_z\}$$

$$\hat{J}^2|j,m\rangle = j(j+1)|jm\rangle \qquad \hat{J}_z|j,m\rangle = m|j,m\rangle$$

vector picture

For given j $-j \le m \le j$ in integer steps
j is half-integer or whole integer

Raising and lowering
$$\hat{J}_\pm|j,m\rangle = \sqrt{j(j+1) \mp m(m\pm 1)}\,|j, m\pm 1\rangle$$

What group are we talking about?
We must be able to accommodate both
½-integer and full-integer j.

Consider $j = \frac{1}{2}$: $\quad \hat{D}^{(\frac{1}{2})}(\phi, \vec{n}) = e^{-i\phi \, \vec{n} \cdot \frac{\hat{\vec{\sigma}}}{2}}$

$$\vec{J} = \vec{S} = \frac{\vec{\sigma}}{2}$$

Consider a sequence of π rotations

$$\hat{D}^{(\frac{1}{2})}(\pi, \vec{n}) \; \hat{D}^{(\frac{1}{2})}(\pi, \vec{n}) = e^{-i \, 2\pi \, \vec{n} \cdot \frac{\vec{\sigma}}{2}}$$

Aside: ~~$\cancel{}$~~ Recall $e^{-i\frac{\phi}{2} \vec{n} \cdot \vec{\sigma}} = \cos(\frac{\phi}{2}) \hat{\mathbb{1}} - i \sin(\frac{\phi}{2}) \vec{n} \cdot \hat{\vec{\sigma}}$

$$\Rightarrow e^{-i \, 2\pi \, \vec{n} \cdot \hat{\vec{\sigma}}/2} = \cos(\frac{2\pi}{2}) \hat{\mathbb{1}} - i \sin(\frac{2\pi}{2}) \vec{n} \cdot \hat{\vec{\sigma}}$$

$$= -\hat{\mathbb{1}}$$

$$\Rightarrow \hat{D}^{(\frac{1}{2})}(\pi, \vec{n}) \; \hat{D}^{(\frac{1}{2})}(\pi, \vec{n}) = -\hat{\mathbb{1}} \quad \overset{?}{!}$$

But for $SO(3)$ $\quad \mathbb{R}(\pi, \vec{n}) \, \mathbb{R}(\pi, \vec{n}) = \mathbb{R}(2\pi, \vec{n})$

$$= \hat{\mathbb{1}} \quad !$$

thus, the matrices $\hat{D}^{(\frac{1}{2})}(\vec{n}, \phi)$

<u>do not</u> form a representation of $SO(3)$

So for spin angular momentum this is **not** a representation of SO(3).

The negative sign is an overall phase factor to the wave function which has no physical consequence

$$D^{(\frac{1}{2})}(2\pi) \, |\psi\rangle = -1 \, |\psi\rangle \equiv |\psi\rangle$$

So, what group describes rotations in Q.M.?

Consider spin-$\frac{1}{2}$ $\quad \hat{U}(\vec{n}\phi) = e^{-i\frac{\phi}{2}\hat{\sigma}_n} = \cos\frac{\phi}{2}\hat{1} - i\sin\frac{\phi}{2}\hat{\sigma}_n$

$$\hat{\sigma}_n = \vec{e}_n \cdot \hat{\vec{\sigma}}$$

Matrix in basis $\{|\pm\rangle_z\}$

$$D^{(\frac{1}{2})} = \begin{bmatrix} \cos\frac{\phi}{2} - i\sin\frac{\phi}{2} n_z & -i\sin\frac{\phi}{2}(n_x - in_y) \\ -i\sin\frac{\phi}{2}(n_x + in_y) & \cos\frac{\phi}{2} + i\sin\frac{\phi}{2}n_z \end{bmatrix}$$

Of the form $\quad D^{(\frac{1}{2})} = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \qquad |a|^2 + |b|^2 = 1$

Group 2x2, unitary, det=1 matrices $\qquad SU(2)$

✸ Loosely speaking $SU(2)$ is like a "doubled" ~~implementation~~ implementation of $SO(3)$.

$$\hat{U}^\dagger(\vec{n}\phi) \langle\psi| \hat{\sigma}_i |\psi\rangle \hat{U}(\vec{n}\phi) = R_{ij} \langle\psi|\sigma_j|\psi\rangle$$

ϕ and $\phi + 2\pi$ yield the same

R_{ij}

The rotation group in quantum mechanics is thus SU(2). The matrix representation for $j = \frac{1}{2}$ is the "defining representation".

The Lie algebra for SU(2) are generated by the spin-$\frac{1}{2}$ matrices $\vec{S} = \frac{\vec{\sigma}}{2}$

$$\hat{S}_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{S}_2 = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\hat{S}_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

With defining commutators $[\hat{S}_i, \hat{S}_j] = i\,\epsilon_{ijk}\hat{S}_k$

The matrices $\hat{D}^{(j)}$ of dim $(2j+1) \times (2j+1)$ form all the possible <u>irreducible</u> <u>representations</u> of SU(2)

• Thus, there is a $j=0$ irrep. This is the trivial group with all elements equal to $\mathbb{1}$
• There is a $j=1$ irrep, a $j=\frac{3}{2}$ irrep, etc.

<u>Note</u>: The Lie algebra for SU(2) is the same as the Lie algebra for SO(3)

Both of these manifold look <u>locally</u> the same. But globally they are different (see supplement)

Euler Angle Decomposition

Wigner gave a description for how to calculate the rotation matrices explicitly. Instead of using an axis-angle decomposition, he used an Euler-angle decomposition. As is well known from the classical theory of rigid-body motion, an arbitrary rotation can be written as a sequence of rotations about a space fixed z-axis and y-axis.

$$\hat{D}(\alpha, \beta, \gamma) \equiv \hat{D}_z(\alpha) \, \hat{D}_y(\beta) \, \hat{D}_z(\gamma)$$

Since $\hat{D}_z(\phi) |j, m\rangle = e^{-i\phi \hat{J}_z} |j, m\rangle = e^{-im\phi} |j, m\rangle$

$$D^{(j)}_{m'm}(\alpha, \beta, \gamma) = \langle j m' | \hat{D}_z(\alpha) \, \hat{D}_y(\beta) \, \hat{D}_z(\gamma) | j\, m \rangle$$

$$= e^{-i(m'\alpha + m\gamma)} \underbrace{\langle j m' | e^{-i\beta \hat{J}_y} | j m \rangle}_{\text{|||}}$$

These are known as the "Wigner-d matrices" $\Longrightarrow d^{(j)}_{m'm}$ and are tabulated in many places. Once we have the Wigner-d matrices, we can calculate an arbitrary rotation by multiplying the matrix elements by appropriate phases according to the Euler angles α and γ