Lecture :19 Clebsch-Gordan and Addition of Angular Momentum

The coupling together of angular momenta and the relevant representations of the rotation group are of central importance in the study of tensors in quantum mechanics — our goal for this unit.

Recall : Addition of angular momentum

Consider two angular momenta $\vec{J}_1$ and $\vec{J}_2$ each acting on a Hilbert space h_1 and h_2 (each orbit and spin).

We define the joint Hilbert space $\mathcal{H} = h_1 \otimes h_2$

The total angular momentum acting on $\mathcal{H}$

$$\hat{\vec{J}} = \hat{\vec{J}}_1 \otimes \hat{\mathbb{1}}_2 + \hat{\mathbb{1}}_1 \otimes \hat{\vec{J}}_2 = \vec{J}_1 + \vec{J}_2$$

The Hilbert space decomposes as

$$\mathcal{H} = \bigoplus_{j_1 j_2} h_1^{(j_1)} \otimes h_2^{(j_2)}$$

Where $h_1^{(j_1)}$ is spanned by $|j_1, m_1\rangle$ $(2j_1+1)$-dim

$h_2^{(j_2)}$ is spanned by $|j_2, m_2\rangle$ $(2j_2+1)$-dim

Uncoupled basis $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$

The "coupled-basis" is defined as simultaneous eigenstates of $\{\hat{\vec{J}}^2, \hat{J}_z, \hat{j}_1^{\,2}, \hat{j}_2^{\,2}\}$

$\{|J\,M\,j_1\,j_2\rangle\}$ $2J+1$ -dim for fixed value of J, j_1, j_2

Where $\hat{\vec{J}}^2 |J\,M\,j_1\,j_2\rangle = J(J+1)\,|J\,M\,j_1\,j_2\rangle$

$J_z\,|J\,M\,j_1\,j_2\rangle = M\,|J\,M\,j_1\,j_2\rangle$

For given values of j_1 and j_2, the total angular momentum quantum number satisfies the "triangle inequality"

$$|j_1 - j_2| < J \leq j_1 + j_2$$

Thus, we have an alternative decomposition of the total Hilbert space as

$$\mathcal{H} = \bigoplus_{j_1, j_2} \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} \mathcal{H}^{(J)}$$

where $\mathcal{H}^{(J)}$ is spanned by $\{|J\,M_J\,j_1\,j_2\rangle\}$

Note $\displaystyle\sum_{J=|j_1-j_2|}^{j_1+j_2} (2J+1) = (2j_1+1)(2j_2+1)$

As expected

Let us consider the subspace determined by fixed common ~~eigenvectors~~ eigenvalues j_1 and j_2

We have two different completness relations

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 m_1 j_2 m_2\rangle\langle j_1 m_1 j_2 m_2| = \hat{\mathbb{1}}$$

$$\sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-J}^{J} |JM j_1 j_2\rangle\langle JM j_1 j_2| = \hat{\mathbb{1}}$$

The change of basis relation is a great importance

$$|JM j_1 j_2\rangle = \sum_{m_1 m_2} |j_1 m_1 j_2 m_2\rangle\langle j_1 m_1 j_2 m_2 | JM j_1 j_2\rangle$$

$$\equiv \sum_{m_1 m_2} \underbrace{\langle j_1 m_1 j_2 m_2 | JM\rangle} \quad |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$

$$\boxed{\begin{array}{c}\text{"vector addition coefficient"} \\ \text{or} \\ \text{Clebsch-Gordan coefficient}\end{array}}$$

In writing the C-G coefficient, we do not put j_1 j_2 in the ket $|JM\rangle$ since they are common eigenvalues

Selection rules

Since $\hat{J}_z = \hat{j}_{1z} + \hat{j}_{2z}$

$$\langle j_1 m_1\, j_2 m_2 | \hat{J}_z | JM\rangle = M\langle j_1 m_1\, j_2 m_2 | JM\rangle$$

$$= (m_1+m_2)\langle j_1 m_1\, j_2 m_2 | JM\rangle$$

$$\Rightarrow \boxed{\langle j_1 m_1\, j_2 m_2 | JM\rangle = 0 \quad \text{unless} \quad M = m_1+m_2}$$

Also C-G vanishes if J does not satisfy the "triangle inequality": $|j_1 - j_2| \leq J \leq j_1 + j_2$

Other properties of C-G coefficients

- Phase convention: C-G are defined to be real

$$\langle j_1 m_1\, j_2 m_2 | JM\rangle = \langle JM | j_1 m_1\, j_2 m_2\rangle$$

- Normalization: Using completeness we see

$$\sum_{m_1 m_2} |\langle JM | j_1 m_1\, j_2 m_2\rangle|^2 = \sum_{JM} |\langle JM | j_1 m_1\, j_2 m_2\rangle|^2 = 1$$

- Exchange: $\langle JM | j_1 m_1\, j_2 m_2\rangle = (-1)^{J-j_1-j_2}\langle JM | j_2 m_2\, j_1 m_1\rangle$

Note: Sometimes these minus signs are removed by defining so-called 3J-symbols

Finding the C-G coefficients

These days, the are many online calculators for finding the C-G coefficients (including, e.g., Mathematica). Within the subspace where $J = j_1 + j_2$, we can use simple angular momentum algebra

We define the "stretched states" such that

$$J = j_1 + j_2, \qquad M = \pm J = m_1 + m_2$$

$$\Rightarrow \begin{matrix} m_1 = j_1 \\ m_2 = j_2 \end{matrix} \quad \text{or} \quad \begin{matrix} m_1 = -j_1 \\ m_2 = -j_2 \end{matrix}$$

$$\Rightarrow |J \pm J\rangle = |j_1 \pm j_1\rangle \otimes |j_2 \pm j_2\rangle$$

Consider the lowering operator: $\hat{J}_- = \hat{j}_{1-} + \hat{j}_{2-}$

Recall: $\hat{J}_- |J\, M\rangle = \sqrt{J(J+1) - M(M-1)}\, |J\; M-1\rangle$

$$\hat{J}_+ |J\, M\rangle = \sqrt{J(J+1) - M(M+1)}\, |J\; M+1\rangle$$

$$\hat{J}_+ = \hat{J}_-^\dagger$$

$\therefore$ We have recursion relations

$$\hat{J}_- |J, J\rangle = \sqrt{J(J+1) - J(J-1)}\, |J, J-1\rangle$$

$$= \sqrt{j_1(j_1+1) - j_1(j_1-1)}\, |j_1, j_1-1\rangle \otimes |j_2\, j_2\rangle$$

$$+ \sqrt{j_2(j_2+1) - j_2(j_2-1)}\, |j_1, j_1\rangle \otimes |j_2\, j_2-1\rangle$$

Example: Addition of two spin-$\frac{1}{2}$ particles.

Uncoupled basis: $|s=\frac{1}{2}, m_s\rangle \otimes |s=\frac{1}{2}, m_s'\rangle \qquad \begin{matrix} m_s \\ m_s' \end{matrix} = \pm\frac{1}{2}$

Short hand for uncoupled basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$

To find the coupled basis, start with the stretched state

$$|S=1, M_s=1\rangle = |s=\tfrac{1}{2}, m_s=\tfrac{1}{2}\rangle \otimes |s=\tfrac{1}{2}, m_s=\tfrac{1}{2}\rangle \boxed{= |\uparrow\uparrow\rangle = |11\rangle}$$

Apply $\hat{S}_- = \hat{S}_{1-} \otimes \hat{\mathbb{1}}_2 + \hat{\mathbb{1}}_1 \otimes \hat{S}_{2-}$

$$\hat{S}_- = \hat{\sigma}_- \quad \text{so} \quad \hat{\sigma}_-|\uparrow\rangle = |\downarrow\rangle \qquad \hat{\sigma}_-|\downarrow\rangle = 0$$

$$\therefore \hat{S}_-|1,1\rangle = \sqrt{1(1+1) - 1(1-1)}\,|1,0\rangle = \sqrt{2}\,|1,0\rangle$$

$$= (\hat{\sigma}_{1-}|\uparrow\rangle) \otimes |\uparrow\rangle + |\uparrow\rangle \otimes (\hat{\sigma}_{2-}|\uparrow\rangle)$$

$$= |\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle$$

$$\Rightarrow \boxed{|10\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)}$$

The state $\boxed{|1\,{-1}\rangle = |\downarrow\downarrow\rangle}$ (the other streched state)

The states
$$\boxed{\begin{aligned} |11\rangle &= |\uparrow\uparrow\rangle \\ |10\rangle &= \tfrac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ |1\,{-1}\rangle &= |\downarrow\downarrow\rangle \end{aligned}}$$
Form the triplet

Note: The states in the subspace with $J = J_{Max} = j_1 + j_2$ are symmetric w.r.t. exchange of j_1 and j_2, since we start with the stretch state and the apply the lowering operator, $\hat{J}_- = \hat{j}_{1-} + \hat{j}_{2-}$.

For two spin-$\frac{1}{2}$ particles, $S_{max} = \frac{1}{2} + \frac{1}{2} = 1$.
The $2S_{max} + 1 = 3$ different symmetric states

$$\left\{ |\uparrow\uparrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), |\downarrow\downarrow\rangle \right\}$$

The only other possible value of $S = |\frac{1}{2} - \frac{1}{2}| = 0$

The state $|0, 0\rangle$ is a superposition of $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ and must be orthogonal to all states with $S=1$

thus, $$|0\,0\rangle = c_a |\uparrow\downarrow\rangle + c_b |\downarrow\uparrow\rangle$$

Require $\langle 1\, M | 0\, 0\rangle = 0$, (trivial for $M = \pm 1$)

For $M=0 \Rightarrow c_a + c_b = 0 \Rightarrow c_a = -c_b$

Normalized, (up to an arbitrary phase)

$$\boxed{|0\,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)}$$

Singlet: anti-symmetric w.r.t. exchange of $\vec{S}_1$ and $\vec{S}_2$

Rotations in Uncoupled and Coupled Basis

For a given generator of angular momentum $\hat{\vec{j}}$, the matrices $D^{(j)}_{m'm} = \langle j m' | e^{-i\theta \vec{n}\cdot\hat{\vec{j}}} | j m \rangle$ form irreps of $SU(2)$ as $(2j+1)\times(2j+1)$ matrices

Consider now rotations generated by $\hat{\vec{J}} = \hat{\vec{j}}_1 + \hat{\vec{j}}_2$

$$e^{-i\theta\vec{n}\cdot\hat{\vec{J}}} = e^{-i\theta\vec{n}\cdot(\hat{\vec{j}}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \hat{\vec{j}}_2)}$$

$$= e^{-i\theta\vec{n}\cdot\hat{\vec{j}}_1} \otimes e^{-i\theta\vec{n}\cdot\vec{j}_2}$$

Thus we have a representation of the rotation matrices on $\mathcal{H} = h_1^{(j_1)} \otimes h_2^{(j_2)}$ via the uncouped basis

$$\langle j_1 m_1' \, j_2 m_2' | e^{-i\theta\vec{n}\cdot\hat{\vec{J}}} | j_1 m_1 \, j_2 m_2 \rangle = D^{(j_1)}_{m_1' m_1} \otimes D^{(j_2)}_{m_2' m_2}$$

<u>Example:</u> Two spin-1/2 particles $D^{(\frac{1}{2})}_{(\theta,\vec{n})} = \cos\frac{\theta}{2}\,\mathbb{1} \, i\sin\frac{\theta}{2}\,\vec{n}\cdot\hat{\vec{\sigma}}$

Consider a rotation by π about the y-axis $\Rightarrow D^{(\frac{1}{2})} = -i\hat{\sigma}_y$

$$\Rightarrow D^{(\frac{1}{2})} \otimes D^{(\frac{1}{2})} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

<u>Clearly reducable</u>

bases $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$

Example: Action on $|10\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

$$D_{y}^{(\frac{1}{2})}(\pi) \otimes D_{y}^{(\frac{1}{2})}(\pi)|10\rangle = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = - \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$= -|10\rangle$$

The matrices $D^{(j_1)}_{m_1' m_1} \otimes D^{(j_2)}_{m_2' m_2}$ act symmetrically on the symmetric subspace $\Rightarrow$ invariant subspace $\Rightarrow$ <u>reducible</u>

Consider now the representation in the coupled basis

$$D^{(J)}_{M'M} = \langle J M'_{j_1 j_2}| e^{-i\theta \vec{n}\cdot\hat{\vec{J}}} |J M_{j_1 j_2}\rangle$$

These matrices are <u>irreducible</u>

because are $2J+1$ vectors are coupled

<u>Thus</u>

$$\boxed{\underbrace{D^{(j_1)} \otimes D^{(j_2)}}_{\text{reducible representation}} = D^{(j_1+j_2)} \oplus D^{(j_1+j_2-1)} \oplus \cdots \oplus \underbrace{D^{(|j_1-j_2|)}}_{\text{irreducible}}}$$

Example: Return to rotation by π around y on state $|1,0\rangle$

Irrep: $D_y^{(1)}(\pi) = d^{(1)}(\pi)$ ← Wigner-d matrix

$$d^{(1)}(\beta) = \begin{bmatrix} \frac{1+\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{bmatrix}$$

in basis $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$

$$d^{(1)}(\pi) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ +1 & 0 & 0 \end{bmatrix}$$

$\Rightarrow d^{(1)}(\pi)|1,0\rangle = -|1,0\rangle$ (as before in reducible rep)

$$D_y^{(\frac{1}{2})}(\pi) \otimes D_y^{(\frac{1}{2})}(\pi) = \begin{array}{c} \begin{bmatrix} \overbrace{1}^{D^{(0)}(\pi)} & & & \\ & 0 & 0 & 1 \\ & 0 & -1 & 0 \\ & \underbrace{1 \quad}_{} & \underbrace{0}_{D^{(1)}(\pi)} & \underbrace{\quad 0}_{} \end{bmatrix} \end{array} \begin{array}{l} |0,0\rangle \\ |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \end{array}$$

$$D^{(j_1)} \otimes D^{(j_2)} = \begin{bmatrix} D^{(j_1+j_2)} & & & \\ & D^{(j_1+j_2-1)} & & \\ & & \ddots & \\ & & & D^{|j_1-j_2|} \end{bmatrix}$$

We can relate the matrix elements of the uncoupled representation to that of the coupled representation via a change of basis

$$D^{(j_1)}_{m_1' m_1} \otimes D^{(j_2)}_{m_2' m_2} = \langle j_1 m_1' j_2 m_2' | \, e^{-i\theta \vec{n}\cdot\vec{J}} \, | j_1 m_1 j_2 m_2 \rangle$$

$$= \sum_{\substack{J,M \\ J'M'}} \langle j_1 m_1' j_2 m_2' | J' M' j_1 j_2 \rangle \underbrace{\langle J' M' j_1 j_2 | e^{-i\theta \vec{n}\cdot\vec{J}} | J M j_1 j_2 \rangle}_{= D^{(J)}_{M'M} \, \delta_{JJ'}} \langle J M j_1 j_2 | j_1 m_1 j_2 m_2 \rangle$$

$$\boxed{D^{(j_1)}_{m_1' m_1} \otimes D^{(j_2)}_{m_2' m_2} = \sum_{J=|j_1-j_2|}^{J=j_1+j_2} \sum_{M=-J}^{J} \langle JM | j_1 m_1 j_2 m_2 \rangle \langle JM' | j_1 m_1' j_2 m_2' \rangle \, D^{(J)}_{M'M}}$$

"Clebsch-Gordan series"