Lecture 25: Fermi's Golden Rule

Coherent vs. Incoherent evolution

We have seen that for a perfectly monochromatic driving force at ω_L near-resonant with two-levels $|g\rangle$ and $|e\rangle$ separated by frequency ω_{eg}, the transition probability is given by Rabi's formula

$$P_{e\leftarrow g}(t) = \frac{\Omega^2}{\tilde{\Omega}^2}\sin^2\left(\frac{\tilde{\Omega}t}{2}\right) \quad \text{where } \Omega = \frac{\vec{d}_{eg}\cdot\vec{E}_0}{\hbar}$$

$$\Delta \equiv \omega_L - \omega_{eg}, \quad \tilde{\Omega} = \sqrt{\Omega^2+\Delta^2}$$

This solution depicts an oscillation of the system between ground and excited states and is thus <u>coherent</u>. At intermediate times the system is in a quantum superposition of $|e\rangle$ and $|g\rangle$

$$|\psi(t)\rangle = \left(\cos\left(\frac{\tilde{\Omega}t}{2}\right) + i\frac{\Delta}{\tilde{\Omega}}\sin\left(\frac{\tilde{\Omega}t}{2}\right)\right)|g\rangle + i\,\frac{\Omega}{\tilde{\Omega}}\sin\left(\frac{\tilde{\Omega}t}{2}\right)|e\rangle$$

This is a <u>very</u> special situation. It requires

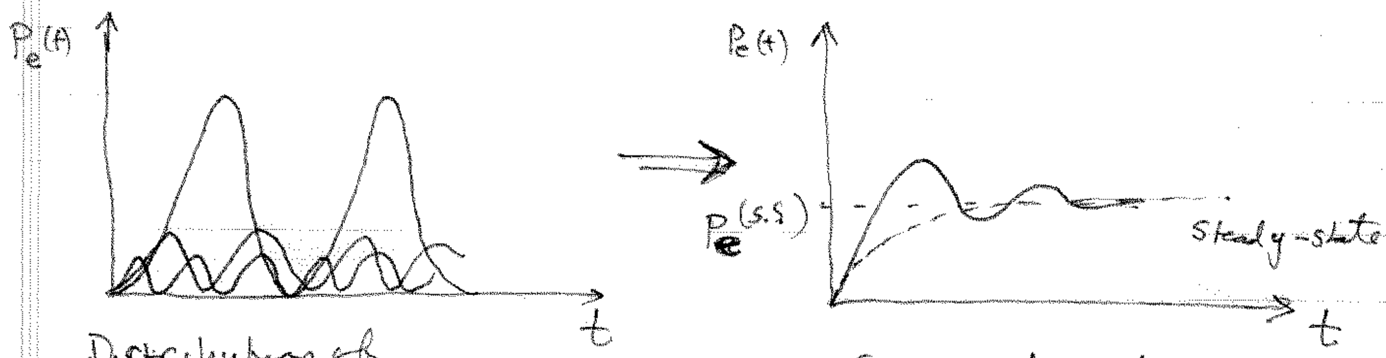
- Very stable driving frequency $\Rightarrow$ narrow "linewidth" of field
- Very well defined energy levels $\Rightarrow$ narrow "linewidth" of transition $|g\rangle \to |e\rangle$

In contrast, suppose the field does not have a well defined frequency, but instead has some spread in frequencies over some bandwith $\Delta\omega$. An approximate transition rate would then average over the resulting detunings:

$$P_{e\leftarrow g}(t) = \int_{-\infty}^{\infty} d\omega_L \, \mathcal{D}(\omega_L) \left(\frac{\Omega^2}{\Omega^2 + (\omega_L - \omega_{eg})^2} \right) \sin^2\left(\frac{\sqrt{\Omega^2 + (\omega_L-\omega_{eg})^2}}{2} t \right)$$

distribution
"density of modes"

This amounts to a weighted sum of different Rabi floppings at different $\tilde{\Omega} = \sqrt{\Omega^2 + \Delta^2}$



Distribution of Rabi-frequencies

Sum = damped Rabi oscillations

We see that the resulting evolution is a damped oscillation. It reaches a steady-state and is "incoherent"

Note: The same result would follow for a distribution of atomic frequency $\mathcal{D}(\omega_{eg})$ if the transition $|e\rangle \to |g\rangle$ has a finite "linewidth"

Fermi's Golden Rule

When the evolution of the system is incoherent and the driving force is "weak", we can return to perturbation theory to get an approximate and useful expression for the transition probability.

Recall for a single driving frequency ω_D and system resonance ω_{eg}, we had (absorption)

$$P_{f\leftarrow i}(\Delta) = \frac{|\langle u_f|\hat{H}^{(+)}_{(\omega_D)}|u_i\rangle|^2}{\hbar^2} \frac{4\sin^2(\frac{\Delta T}{2})}{\Delta^2}$$

Now let us average over a distribution $\mathcal{D}$ (this could be $\mathcal{D}(\omega_D)$ or $\mathcal{D}(\omega_{eg})$ as we will see)

$$\Rightarrow P_{f\leftarrow i} = \int_{-\infty}^{\infty} d\omega\, \mathcal{D}(\omega)\; P_{f\leftarrow i}(\Delta) \qquad \text{where } \omega = \omega_D \text{ or } \omega_{eg}$$

$$= \int_{-\infty}^{\infty} d\omega\, \mathcal{D}(\omega)\, \frac{|\langle u_f|\hat{H}^{(+)}_{(\omega_D)}|u_i\rangle|^2}{\hbar^2} \frac{4\sin^2(\frac{\Delta T}{2})}{\Delta^2}$$

We consider here the <u>incoherent</u> <u>limit</u> so that the characteristic bandwidth of the distribution $\Delta\omega$ is very large. Specifically, we assume

$$\Delta\omega \gg \frac{1}{T}$$

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In this limit, $\frac{\sin^2(\Delta T/2)}{\Delta^2}$ looks like a delta function w.r.t. $\mathcal{D}(\omega)$. This means we can take the $T \to \infty$ limit since $T >> \frac{1}{\Delta\omega} \to 0$ as $\Delta\omega \to \infty$.

Of course $T \to \infty$ only makes sense here if $P_{f\leftarrow i}(T) << 1$, which must be checked for consistency

Aside: $\lim\limits_{a\to\infty} \frac{\sin^2(ax)}{a(x)^2} = \pi\,\delta(x)$

$$\Rightarrow \lim_{\text{"}T\to\infty\text{"}} \frac{\sin^2\left(\frac{\Delta T}{2}\right)}{\Delta^2} = \frac{\pi T}{2}\,\delta(\Delta) = \frac{\pi T}{2}\,\delta(\omega_f - (\omega_i + \omega_D))$$

- $$P_{f\leftarrow i} \cong T\int_{-\infty}^{\infty} d\omega\, \mathcal{D}(\omega)\, \frac{2\pi}{\hbar} |\langle u_f|\, \hat{H}^{(+)}_{(\omega_D)}\, |u_i\rangle|^2\, \delta(E_f - (E_i + \hbar\omega_D))$$

The transition probability thus increases linearly with time.

We thus define the transition rate

$$\boxed{W_{f\leftarrow i} \equiv \frac{dP_{f\leftarrow i}}{dT} = \int_{-\infty}^{\infty} d\omega\, \mathcal{D}(\omega)\; W_{f\leftarrow i}(\omega) \qquad \left(\omega = \omega_{fi} \text{ or } \omega_D\right)}$$

$$W_{f\leftarrow i}(\omega) = \frac{2\pi}{\hbar} |\langle u_f|\, \hat{H}^{(+)}_{(\omega_D)}\, |u_i\rangle|^2\, \delta(E_f - (E_i + \hbar\omega_D))$$

The boxed result is known as "Fermi's Golden Rule". It is a great importance because of its wide applicability. However its applicability is not universal. It requires incoherent excitation

$$\underline{\Delta\omega \gg \frac{1}{T}}$$

Other less restrictive conditions are

(i) Weak field (first-order perturbation)

This is satisfied if $|\langle u_f|\hat{H}^{(1)}|u_i\rangle| \ll \hbar\Delta\omega$

For E/M field $\Rightarrow$ $\Omega_1 \ll \Delta\omega$ (small Rabi frequency)

$\Rightarrow$ No saturation

(ii) $P_{f\leftarrow i} \ll 1$ $\Rightarrow$ "Short T"

$$\Rightarrow T \ll (W_{f\leftarrow i})^{-1} \sim 2\pi \frac{\hbar\Delta\omega}{|\langle u_f|\hat{H}^{(1)}|u_i\rangle|^2}$$

However we also assumed $T \gg \frac{1}{\Delta\omega}$

So again we must have $\frac{\Delta\omega}{|\langle u_f|\hat{H}^{(1)}|u_i\rangle} \ll 1$

Note: We will see that we can relax the "Short T" condition for incoherent evolution.

Broadband excitation

Suppose our two-level system is driven by a lamp, rather than a laser. The field has a spectral energy density $\mathcal{U}(\omega) = \frac{\text{energy density}}{\text{frequency interval}}$

so that $\int d\omega\, \mathcal{U}(\omega) = $ total energy density

If $\vec{E}^{(+)}_{(\omega)}$ is the positive frequency electric field at ω

$$\mathcal{U}(\omega) = \frac{1}{2\pi} |\vec{E}^{(+)}(\omega)|^2 \underbrace{\mathcal{D}(\omega)}_{\text{distribution}}$$

Now according to Fermi's Golden Rule

$$W_{e\leftarrow g} = 2\pi \frac{|\vec{d}_{eg} \cdot \vec{E}^{(+)}(\omega_{eg})|^2}{\hbar^2} \mathcal{D}(\omega = \omega_{eg})$$

Aside: The lamp is unpolarized $\Rightarrow \frac{1}{3}$ of energy density along $\vec{d}_{eg}$

$$\Rightarrow W_{e\leftarrow g} = \frac{2\pi}{3} \frac{|\vec{d}_{eg}|^2}{\hbar^2} |\vec{E}^{(+)}_{(\omega_{eg})}|^2 \mathcal{D}(\omega = \omega_{eg}) =$$

$$\Rightarrow \boxed{W_{e\leftarrow g} = \frac{4\pi^2}{3} \frac{|\vec{d}_{eg}|^2}{\hbar^2} \mathcal{U}(\omega = \omega_{eg})}$$

The coefficient in front of $\mathcal{U}$ is known as the Einstein B coefficient

$$\boxed{W_{e\leftarrow g} = B\, \mathcal{U}(\omega_{eg}) \qquad B = \frac{4\pi^2}{3} \frac{|\vec{d}_{eg}|}{\hbar^2}}$$

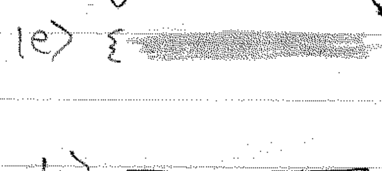
Note $w_{e\leftarrow g} = W_{g\leftarrow e}$ for two-levels

$\Rightarrow$ Rate of absorption $\equiv$ Rate of <u>induced</u> emission

The induced emission is known as <u>stimulated emission</u>

- <u>Narrow-band excitation and natural linewidth</u>

Let us return to laser-like excitation. We will not generally see ~~[illegible]~~ perfect Rabi oscillation lasting indefinitely because of the effect of finite atomic linewidth. The excited state is not a perfectly discrete level due to a "natural linewidth"



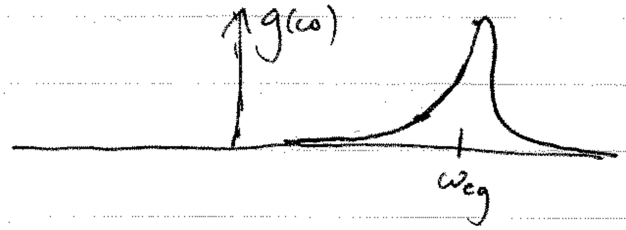
Physically the linewidth arise due to coupling of the excited state to the quantum fluctuations of the electromagnetic field. This noisy bath induces <u>spontaneous</u> emission $|e\rangle \to |g\rangle$.
Thus, even in the absence of an inducing field, the state $|e\rangle$ will decay spontaneously to $|g\rangle$ at a rate $\Gamma \equiv A$ (Einstein A coefficient).

$$\frac{1}{\Gamma} = \tau = \text{"lifetime" of } |e\rangle$$

"Time-energy uncertainty"

The "lifetime brodening" of $|e\rangle$ leads to a distribution of "density of states" $|e\rangle$ according to Lorentzian:

$$g(\omega) = \frac{\Gamma/2\pi}{(\omega-\omega_{eg})^2 + \frac{\Gamma^2}{4}} \qquad \int_{-\infty}^{\infty} d\omega \, g(\omega) = 1$$

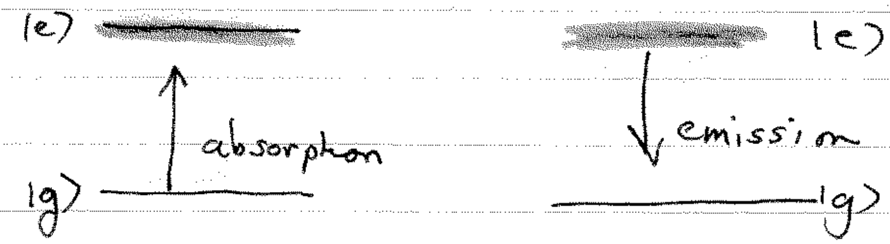


atomic lineshape

By Fermi's Golden rule:

$$W_{e\leftarrow g} = \frac{2\pi}{\hbar^2}\left|\frac{\vec{d}_{eg}\cdot\vec{E}_0}{2}\right|^2 g(\omega_D) = \frac{\Omega^2/4}{\Delta^2+\frac{\Gamma^2}{4}}\,\Gamma = W_{g\leftarrow e}$$

Physical picture: detailed balance



Steady state is established when

$$P_g\,(R_{abs}) = P_e\,(R_{emission})$$

To lowest order in perturbation theory we can neglect stimulated emission $\Rightarrow$ $R_{emission} = \Gamma$

$$\Rightarrow \frac{\Omega^2/4}{\Delta^2+\frac{\Gamma^2}{4}} = \text{steady state population in } |e\rangle$$