The case of degeneracies

Suppose for a given eigenvalue λ of a normal operator $\hat{M}$, there are g_λ different eigenvectors (different means not parallel to one another) $\{|u_\lambda^{(i)}\rangle \mid i=1,2,\ldots,g_\lambda\}$

$$\Rightarrow \hat{M}|u_\lambda^{(i)}\rangle = \lambda|u_\lambda^{(i)}\rangle$$

Since $\{|u_\lambda^{(i)}\rangle\}$ are linearly independent, they span a subspace of $\mathcal{H}$ of dim g_λ.

$\Rightarrow$ Any vector $|v_\lambda\rangle = \sum_{i=1}^{g_\lambda} \alpha_i |u_\lambda^{(i)}\rangle$ is also an eigenvector of $\hat{M}$ with eigenvalue λ.

$\Rightarrow \{|u_\lambda^{(i)}\rangle\}$ spans an "eigensubspace" $\mathcal{H}_\lambda$

$\Rightarrow$ We can always find an orthormal basis of eigenvectors, by choosing an orthonormal basis for the spaces $\mathcal{H}_\lambda$. Note, vectors in different subspaces, with different values of λ are orthogonal by the arguement above (nondegenerate eigenvectors are orthogonal).

$\Rightarrow$ There always exists an orthonormal basis of eigenvectors $\{|e_\lambda^{(i)}\rangle\}$

$$\langle e_\lambda^{(i)}|e_{\lambda'}^{(j)}\rangle = \delta_{\lambda\lambda'}\,\delta_{ii'}$$

For a given λ $\{e_\lambda^{(i)} \mid i=1,2,..,g_\lambda\}$ spans a subspace $\mathcal{H}_\lambda$ of dimension g_λ.

The different subspaces are orthogonal: $\mathcal{H}_\lambda \perp \mathcal{H}_{\lambda'}$ if $\lambda\neq\lambda'$

We can for a resolution of the identity:

$$\hat{\mathbb{1}} = \sum_\lambda \sum_{i=1}^{g_\lambda} |e_\lambda^{(i)}\rangle\langle e_\lambda^{(i)}| = \sum_\lambda \hat{\mathbb{P}}_\lambda$$

$\hat{\mathbb{P}}_\lambda = \sum_{i=1}^{g_\lambda} |e_\lambda^{(i)}\rangle\langle e_\lambda^{(i)}|$ is a "projection operator" — it acts to "project" a vector onto the subspace $\mathcal{H}_\lambda$. $\hat{\mathbb{P}}_\lambda|\psi\rangle = \sum_{i=1}^{g_\lambda}\langle e_\lambda^{(i)}|\psi\rangle\, |e_\lambda^{(i)}\rangle \in \mathcal{H}_\lambda$

The projection operators satisfies $\hat{\mathbb{P}}_\lambda = \hat{\mathbb{P}}_\lambda^\dagger$ $\quad \hat{\mathbb{P}}_\lambda\hat{\mathbb{P}}_{\lambda'} = \delta_{\lambda\lambda'}$ (orthogonal projection)

The total Hilbert space is said to be the "direct sum" of all the orthogonal eigensubspaces:

$$\mathcal{H} = \underset{\lambda}{\oplus}\, \mathcal{H}_\lambda$$

"direct sum" = union

The matrix representation of $\hat{M}$ is the basis of its eigenvalues has diagonal block

$$\hat{M} = \left[\begin{array}{c|ccc|cc} \lambda_1 & & & & & \\ \hline & \lambda_2 & & & & \\ & & \lambda_2 & & & \\ & & & \lambda_2 & & \\ \hline & & & & \lambda_3 & \\ & & & & & \lambda_3 \end{array}\right] \begin{array}{l} \leftarrow \text{eigenvalue } \lambda_1 \text{: Nondegenerate} \\ \\ \leftarrow \text{eigenvalue } \lambda_2 \text{: triply degenerate} \\ \\ \leftarrow \text{eigenvalue } \lambda_3 \text{: doubly degenerate} \end{array}$$

Zeros in all other elements

Commuting operators and eigenspaces

Consider two Hermitian operators that commute: $[\hat{A}, \hat{B}] = 0$

Suppose $|a\rangle$ is an eigenvector of $\hat{A}$, $\hat{A}|a\rangle = a|a\rangle \Rightarrow \hat{B}|a\rangle$ is an eigenvector of $\hat{A}$ with the same eigenvalue: $\hat{A}(\hat{B}|a\rangle) = \hat{B}(\hat{A}|a\rangle) = a(\hat{B}|a\rangle)$ q.e.d.

If $|a\rangle$ is a nondegenerate eigenvector $\Rightarrow \hat{B}|a\rangle \propto |a\rangle \Rightarrow |a\rangle$ is also an eigenvector of $\hat{B}$, with a nondegenerate eigenvalue b. Thus we can write the vector $|a,b\rangle$ s.t. $\hat{A}|a,b\rangle = a|a,b\rangle$, $\hat{B}|a,b\rangle = b|a,b\rangle$

If $|a\rangle$ is degenerate, then we can say that $\hat{B}|a\rangle \in \mathcal{H}_a$, the eigensubspace spanned by the degenerate eigenvectors with eigenvalue a.

Note: If $|a\rangle$ and $|a'\rangle$ are nondegenerate $\Rightarrow \langle a'|\hat{B}|a\rangle = 0$

Thus, in the basis of eigenvectors of $\hat{A}$, $\{|a\rangle\}$, if $[\hat{A},\hat{B}]=0$, then the representation of $\hat{B}$ will be "block diagonal"

$$\hat{B} \doteq \left[\begin{array}{c|c|c} \hat{B}_1 & & \\ \hline & \hat{B}_2 & \\ \hline & & \hat{B}_3 \end{array}\right] \begin{array}{l} \leftarrow \mathcal{H}_1 \\ \leftarrow \mathcal{H}_2 \\ \leftarrow \mathcal{H}_3 \end{array} = \hat{B}_1 \oplus \hat{B}_2 \oplus \hat{B}_3$$

The operators $\hat{B}_a = \hat{\mathbb{P}}_a \hat{B} \hat{\mathbb{P}}_a$ ($\hat{B}$ projected in subspace $\mathcal{H}_a$) can each be diagonalized. Thus, with a subspace $\mathcal{H}_a$ of degeneracy g_a, there are g_a eigenvectors of $\hat{B}$

$\Rightarrow$ If $[\hat{A}, \hat{B}] = 0$, there exist a set of common eigenvectors of $\hat{A}$ and $\hat{B}$. We often say that $\hat{A}$ and $\hat{B}$ are mutually diagonalizable in the same basis.

Note: If $[\hat{A}, \hat{B}] \neq 0$, it doesn't mean that they can't share any eigenvectors. However they cannot share <u>all</u> eigenvectors. For if they did, they would be diagonal in the same basis, and thus they would commute.

<u>Complete Set of Commuting Operators:</u>

Now after we diagonalize $\hat{B}_a$, it might be the case that some of its eigenvectors are degenerate. In that case, the eigenvector is not uniquely specified by the two eigenvalues a and b. However, this means that there can be a third operator $\hat{C}$ which mutually commutes with $\hat{A}$ and $\hat{B}$: $[\hat{A}, \hat{B}] = [\hat{A}, \hat{C}] = [\hat{C}, \hat{A}] = 0$. In the subspaces $\hat{\mathbb{P}}_{a,b}$, $\hat{C}$ will be block diagonal. We can then diagonalize $\hat{C}_{a,b}$, and keep going until we have no more degeneracy

$\Rightarrow$ A complete set of mutually commuting operators $\{\hat{A}, \hat{B}, \hat{C}, \cdots\}$ is the minimal set of normal operators that all commute with one another such that the common eigenvectors of all the operators are uniquely specified. I.e, there is only one vector $|a,b,c,\cdots\rangle$ (up to multiplication by a scalar) such that

$$\hat{A}|a,b,c,\ldots\rangle = a|a,b,c,\ldots\rangle$$
$$\hat{B}|a,b,c,\ldots\rangle = b|a,b,c,\ldots\rangle$$
$$\hat{C}|a,b,c,\ldots\rangle = c|a,b,c,\ldots\rangle$$

The collection of eigenvalues $\{a,b,c,\ldots\}$ specify the state.