Physics 522, Spring 2016
Problem Set #1
Due: Tuesday Jan. 26, 2016 @ 5PM

Problem 1: The orbital angular momentum operator. (15 points)

The orbital angular momentum operator for a particle with momentum $\hat{\mathbf{p}}$ and position $\hat{\mathbf{x}}$ is, $\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}}$, or in component form $\hat{L}_i = \varepsilon_{ijk}\, \hat{x}_j\, \hat{p}_k$, where *i,j,k* index the Cartesian components and sums go from 1 to 3 (1=x, 2=y,3=z), using the Einstein summation convention.

(a) We know the famous "canonical commutation relations" $\left[\hat{x}_j, \hat{p}_k\right] = i\delta_{jk}\hbar$ (position and momentum for different Cartesian coordinates, otherwise not).

Show that $[\hat{L}_i, \hat{L}_j] = i\hbar\, \varepsilon_{ijk}\, \hat{L}_k$. These are known as the standard SU(2) commutation relations.

(b) Further show some, perhaps less familiar, commutation relations

$$[\hat{L}_i, \hat{x}_j] = i\hbar\varepsilon_{ijk}\, \hat{x}_k, \quad [\hat{L}_i, \hat{p}_j] = i\hbar\varepsilon_{ijk}\, \hat{p}_k, \quad [\hat{L}_i, \hat{r}^2] = [\hat{L}_i, \hat{p}^2] = [\hat{L}_i, \hat{L}^2] = 0 .$$

(c) Prove the uncertainty principle for angular momentum $\Delta J_x \Delta J_y \geq \frac{\hbar}{2}\left|\left\langle \hat{J}_z \right\rangle\right|$, where $\hat{J}_i$ is component of generic angular momentum, orbital or spin.

Problem 2: Spin 1/2 operators and eigenstates (20 points)

A spin 1/2 particle is described by a two dimensional Hilbert space. We typically define a "quantization direction" to be the *z*-direction and define two kets, $\left\{\left|\uparrow_z\right\rangle, \left|\downarrow_z\right\rangle\right\}$, which form an orthonormal basis for the space (called the "standard basis"). The components of the angular momentum operator can then be written

$$\hat{S}_x = \frac{\hbar}{2}\left|\uparrow_z\right\rangle\left\langle\downarrow_z\right| + \frac{\hbar}{2}\left|\downarrow_z\right\rangle\left\langle\uparrow_z\right|, \quad \hat{S}_y = \frac{\hbar}{2i}\left|\uparrow_z\right\rangle\left\langle\downarrow_z\right| - \frac{\hbar}{2i}\left|\downarrow_z\right\rangle\left\langle\uparrow_z\right|, \quad \hat{S}_z = \frac{\hbar}{2}\left|\uparrow_z\right\rangle\left\langle\uparrow_z\right| - \frac{\hbar}{2}\left|\downarrow_z\right\rangle\left\langle\downarrow_z\right| .$$

(a) Find the eigenvalues and eigenvectors of $\left\{\hat{S}_x, \hat{S}_y, \hat{S}_z\right\}$. Are these operators Hermitian and do the eigenvalues/vectors reflect this? Explain.

(b) Express $\left\{\left|\uparrow_z\right\rangle, \left|\downarrow_z\right\rangle\right\}$ in the basis $\left\{\left|\uparrow_x\right\rangle, \left|\downarrow_x\right\rangle\right\}$, the eigenstates of $\hat{S}_x$. Show that the transformation matrix is unitary.

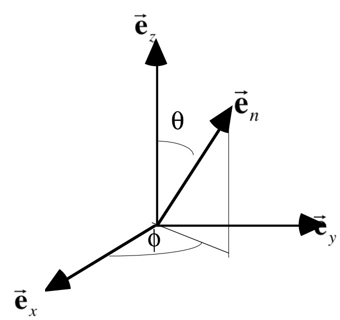
(c) Express $\hat{S}_x, \hat{S}_y, \hat{S}_z$ as outer products in the basis $\{|\uparrow_x\rangle, |\downarrow_x\rangle\}$. Please comment on your results.

(d) Show that the components of spin satisfy $[\hat{S}_i, \hat{S}_j] = i\hbar \varepsilon_{ijk} \hat{S}_k$.

(e) Given a spin-1/2 in the state $|\uparrow_z\rangle$, find ΔS_x and ΔS_y. Show that the uncertainty principle for angular momentum is satisfied for spin as well.

Problem 3: Measurements on a two-state system (15 points)

Given a unit vector $\vec{\mathbf{e}}_n$, defined by angles θ and ϕ with respect to the polar axis *z*,



we can define the ket $|\uparrow_n\rangle = \cos(\theta/2)|\uparrow_z\rangle + e^{i\phi}\sin(\theta/2)|\downarrow_z\rangle$, as the state with spin $+\hbar/2$ along the axis $\vec{\mathbf{e}}_n$.

(a) Show that $\hat{\mathbf{S}} \cdot \vec{\mathbf{e}}_n |\uparrow_n\rangle = \frac{\hbar}{2}|\uparrow_n\rangle$, where $\hat{\mathbf{S}} = \hat{S}_x \vec{\mathbf{e}}_x + \hat{S}_y \vec{\mathbf{e}}_y + \hat{S}_z \vec{\mathbf{e}}_z$, with $\{\hat{S}_x, \hat{S}_y, \hat{S}_z\}$ the three components of the spin 1/2 operator.

Now consider a beam of spin 1/2 atoms that goes through a series of Stern-Gerlach-type measurements as follows:

(i) The first measurement accepts $s_z = +\hbar/2$ and rejects $s_z = -\hbar/2$.

(ii) The second measurement accepts $s_n = +\hbar/2$ and rejects $s_n = -\hbar/2$ (along axis $\vec{\mathbf{e}}_n$).

(iii) The third measurement accepts $s_z = -\hbar/2$ and rejects $s_z = +\hbar/2$.

(b) What is the probability of detecting the final spin with $s_z = -\hbar/2$ given an atom which passes through the first apparatus?

(c) How must we orient the second apparatus if we are to maximize this probability. Please *interpret* your result.