Physics 522, Spring 2016
Problem Set #1
Solutions

Problem 1: The orbital angular momentum operator

(a) $\hat{L}_i = \epsilon_{ijk} \hat{x}_j \hat{P}_k$

$$[\hat{L}_i, \hat{L}_j] = [\epsilon_{iab} \hat{x}_a \hat{P}_b, \epsilon_{jcd} \hat{x}_c \hat{P}_d]$$

$$= \epsilon_{iab} \epsilon_{jcd} [\hat{x}_a \hat{P}_b, \hat{x}_c \hat{P}_d]$$

$$= \epsilon_{iab} \epsilon_{jcd} \Big(\hat{x}_a \hat{x}_c \cancelto{0}{[\hat{P}_b, \hat{P}_d]} + \hat{x}_a [\hat{P}_b, \hat{x}_c] \hat{P}_d + \hat{x}_c [\hat{x}_a, \hat{P}_d] \hat{P}_b + \cancelto{0}{[\hat{x}_a, \hat{x}_c]} \hat{P}_b \hat{P}_d \Big)$$

$$= \epsilon_{iab} \epsilon_{jcd} \left(-i \delta_{bc} \hbar \hat{x}_a \hat{P}_d + i \delta_{ad} \hbar \hat{x}_c \hat{P}_b \right)$$

$$= i\hbar \left(\epsilon_{iab} \epsilon_{jcd} \delta_{ad} \hat{x}_c \hat{P}_b - \epsilon_{iab} \epsilon_{jcd} \delta_{bc} \hat{x}_a \hat{P}_d \right)$$

$$= i\hbar \left(\epsilon_{iab} \epsilon_{jca} \hat{x}_c \hat{P}_b - \epsilon_{iab} \epsilon_{jbd} \hat{x}_a \hat{P}_d \right)$$

(cyclicly permute indices to have first index same)

$$= i\hbar \left(\epsilon_{abi} \epsilon_{ajc} \hat{x}_c \hat{P}_b - \epsilon_{bia} \epsilon_{bdj} \hat{x}_a \hat{P}_d \right)$$

$$\boxed{\text{Now, } \epsilon_{iab} \epsilon_{icd} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}}$$

$$[\hat{L}_i, \hat{L}_j] = i\hbar \Big[(\delta_{bj}\delta_{ic} - \delta_{bc}\delta_{ij})\hat{x}_c \hat{p}_b - (\delta_{id}\delta_{aj} - \delta_{ij}\delta_{ad})\hat{x}_a \hat{p}_d \Big]$$

$$= i\hbar \Big[\hat{x}_i \hat{p}_j - \cancel{\delta_{ij}\hat{x}_b \hat{p}_b} - \hat{x}_j \hat{p}_i + \cancel{\delta_{ij}\hat{x}_a \hat{p}_a} \Big]$$

$$= i\hbar \big[\hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i \big]$$

$$= i\hbar\, \epsilon_{ijk} \hat{L}_k \qquad \square$$

(b) $$[\hat{L}_i, \hat{x}_j] = [\epsilon_{ik\ell}\hat{x}_k \hat{p}_\ell, \hat{x}_j]$$

$$= \epsilon_{ik\ell}\hat{x}_k [\hat{p}_\ell, \hat{x}_j]$$

$$= -i\hbar\, \delta_{\ell j}\epsilon_{ik\ell}\hat{x}_k$$

$$= i\hbar\, \epsilon_{ijk}\hat{x}_k \qquad \square$$

(c) $$[\hat{L}_i, \hat{p}_j] = [\epsilon_{ik\ell}\hat{x}_k \hat{p}_\ell, \hat{p}_j]$$

$$= i\hbar\, \epsilon_{ik\ell}\delta_{kj}\hat{p}_\ell$$

$$= i\hbar\, \epsilon_{ij\ell}\hat{p}_\ell$$ change dummy index $k \to \ell$.

$$\equiv i\hbar\, \epsilon_{ijk}\hat{p}_k \qquad \square$$

$$[\hat{L}_i, \hat{r}^2] = [\epsilon_{ijk} \hat{x}_j \hat{p}_k , \hat{x}_l \hat{x}_l]$$

$$= 2\epsilon_{ijk} \hat{x}_j [\hat{p}_k, \hat{x}_l] \hat{x}_l$$

$$= -2 \epsilon_{ijk} i\hbar \hat{x}_j \hat{x}_k$$

$$= -2i\hbar (\hat{\vec{x}} \times \hat{\vec{x}})_i$$

$$= 0 \quad \square$$

$$[\hat{L}_i, \hat{p}^2] = [\epsilon_{ijk} \hat{x}_j \hat{p}_k , \hat{p}_l \hat{p}_l]$$

$$= 2\epsilon_{ijk} i\hbar \hat{p}_k \hat{p}_j$$

$$= 2i\hbar (\hat{\vec{p}} \times \hat{\vec{p}})_i = 0 \quad \square$$

$$[\hat{L}_i, \hat{L}^2] = [\hat{L}_i, \hat{L}_j \hat{L}_j]$$

$$= [\hat{L}_i, \hat{L}_j] \hat{L}_j + \hat{L}_j [\hat{L}_i, \hat{L}_j]$$

$$= i\hbar (\epsilon_{ijk} \hat{L}_k \hat{L}_j + \epsilon_{ijk} \hat{L}_j \hat{L}_k)$$

(↳ exchange j & k ~~Pick a minus sign~~)

$$= i\hbar (\epsilon_{ijk} \hat{L}_k \hat{L}_j + \epsilon_{ikj} \hat{L}_k \hat{L}_j)$$

$$= i\hbar \hat{L}_k \hat{L}_j \underbrace{(\epsilon_{ijk} + \epsilon_{ikj})}_{=0}$$

$$= 0$$

(c) We know that

$$\Delta A \cdot \Delta B \geqslant \frac{1}{2} \left| \langle [A, B] \rangle \right|$$

$$\text{So } \Delta J_x \Delta J_y \geqslant \frac{1}{2} \left| \langle [J_x, J_y] \rangle \right|$$

$$\geqslant \frac{\hbar}{2} \left| \langle J_z \rangle \right|$$

$$=$$

Problem 2: Spin ½ operators and eigen states.

~~(a)~~ $$\hat{S}_x = \frac{\hbar}{2} \left\{ |\uparrow_z\rangle\langle\downarrow_z| + |\downarrow_z\rangle\langle\uparrow_z| \right\}$$

$$\hat{S}_y = \frac{\hbar}{2i} \left\{ |\uparrow_z\rangle\langle\downarrow_z| - |\downarrow_z\rangle\langle\uparrow_z| \right\}$$

$$\hat{S}_z = \frac{\hbar}{2} \left\{ |\uparrow_z\rangle\langle\uparrow_z| - |\downarrow_z\rangle\langle\downarrow_z| \right\}$$

In Matrix form in the basis $\{ |\uparrow_z\rangle, |\downarrow_z\rangle \}$ (and in units of $\hbar$)

$$\hat{S}_x \doteq \begin{bmatrix} \langle\uparrow_z|\hat{S}_x|\uparrow_z\rangle & \langle\uparrow_z|\hat{S}_x|\downarrow_z\rangle \\ \langle\downarrow_z|\hat{S}_x|\uparrow_z\rangle & \langle\downarrow_z|\hat{S}_x|\downarrow_z\rangle \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\hat{S}_y \doteq \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{S}_z \doteq \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Recall the Pauli Matrices are $\hat{\sigma}_i = 2 \hat{S}_i$

(a)

Eigenvalues and Eigen vectors

$\hat{S}_x$: $\det(\hat{S}_x - \lambda_x \hat{\mathbb{1}}) = \det\begin{bmatrix} -\lambda_x & 1/2 \\ 1/2 & -\lambda_x \end{bmatrix} = \lambda_x^2 - \frac{1}{4} = 0$

$$\boxed{\lambda_x = \pm\frac{1}{2}}$$

- $\lambda_x = +\frac{1}{2} \Rightarrow S_x|\uparrow_x\rangle = \frac{1}{2}|\uparrow_x\rangle$

$$\begin{bmatrix} 0 & \frac{1}{2} \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} a_x^{(+)} \\ b_x^{(+)} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a_x^{(+)} \\ b_x^{(+)} \end{bmatrix}$$

$$\frac{1}{2} b_x^+ = \frac{1}{2} a_x^+ \Rightarrow a_x^+ = b_x^+$$

So $|\uparrow_x\rangle = a_x^+ (|\uparrow_z\rangle + |\downarrow_z\rangle)$

Normalize : $2 a_x^{+2} = 1 \Rightarrow |a_x^+| = \frac{1}{\sqrt{2}}$

Normalized eigenvector is $\boxed{|\uparrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle)}$

- $\lambda_x = -\frac{1}{2} \Rightarrow \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} a_x^- \\ b_x^- \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} a_x^- \\ b_x^- \end{bmatrix}$

$a_x^- = -b_x^- \Rightarrow |\downarrow_x\rangle = a_x^- (|\uparrow_z\rangle - |\downarrow_z\rangle)$

Normalized : $\boxed{|\downarrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle - |\downarrow_z\rangle)}$

$\hat{S}_y$: $\det(\hat{S}_y - \lambda_y \mathbb{1}) = \det\begin{pmatrix} -\lambda_y & -i/2 \\ i/2 & -\lambda_y \end{pmatrix}$

$$= \lambda_y^2 - \frac{1}{4} = 0$$

$$\lambda_y = \pm\frac{1}{2}$$

- $\lambda_y = +\frac{1}{2}$: $\begin{bmatrix} 0 & -i/2 \\ i/2 & 0 \end{bmatrix}\begin{bmatrix} a_y^+ \\ b_y^+ \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a_y^+ \\ b_y^+ \end{bmatrix}$ $\quad a_y^+ = -i\, b_y^+$

$$\Rightarrow \boxed{|\uparrow_y\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle - i\,|\downarrow_z\rangle\right)}$$

- $\lambda_y = -\frac{1}{2}$ $\begin{bmatrix} 0 & -i/2 \\ i/2 & 0 \end{bmatrix}\begin{bmatrix} a_y^- \\ b_y^- \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} a_y^- \\ b_y^- \end{bmatrix} =$ $\quad a_y^+ = i\, b_y^-$

$$\boxed{|\downarrow_y\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + i\,|\downarrow_z\rangle\right)}$$

$\hat{S}_z$: is already diagonal. So eigen vectors are $|\uparrow_z\rangle$ & $|\downarrow_z\rangle$ with eigen values $\pm\frac{1}{2}$.

These operators are hermitian since $\hat{S}_i^\dagger = \hat{S}_i$

The fact that operators are hermition is reflected in the fact that the eigenvectors are orthogonal and the eigenvalues are real

(b)

$$|\uparrow_z\rangle = |\uparrow_x\rangle\langle\uparrow_x|\uparrow_z\rangle + |\downarrow_x\rangle\langle\downarrow_x|\uparrow_z\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle + |\downarrow_x\rangle\right)$$

$$|\downarrow_z\rangle = |\uparrow_x\rangle\langle\uparrow_x|\downarrow_z\rangle + |\downarrow_x\rangle\langle\downarrow_x|\downarrow_z\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle - |\downarrow_x\rangle\right)$$

Transformation matrix

$$U \doteq \begin{bmatrix} \langle\uparrow_x|\uparrow_z\rangle & \langle\uparrow_x|\downarrow_z\rangle \\ \langle\downarrow_x|\uparrow_z\rangle & \langle\downarrow_x|\downarrow_z\rangle \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$U^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = U$$

$$UU^\dagger = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{1} = U^\dagger U$$

U is unitary

(c) Express $\hat{S}_x, \hat{S}_y, \hat{S}_z$ in basis $\{|\uparrow_x\rangle, |\downarrow_x\rangle\}$

we just showed that the transformation matrix is $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$\therefore \quad \hat{S}_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} 1/\sqrt{2}$$

$$= \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix} \Rightarrow \hat{S}_x = \frac{1}{2}\left(|\uparrow_x\rangle\langle\uparrow_x| - |\downarrow_x\rangle\langle\downarrow_x|\right)$$

$$\hat{S}_y \doteq \begin{bmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{bmatrix}$$

$$\Rightarrow \hat{S}_y = \frac{i}{2}\left\{ |\uparrow_x\rangle\langle\downarrow_x| - |\downarrow_x\rangle\langle\uparrow_x| \right\}$$

$$\hat{S}_z = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

$$\hat{S}_z = \frac{1}{2}\left\{ |\uparrow_x\rangle\langle\downarrow_x| + |\downarrow_x\rangle\langle\downarrow_x| \right\}$$

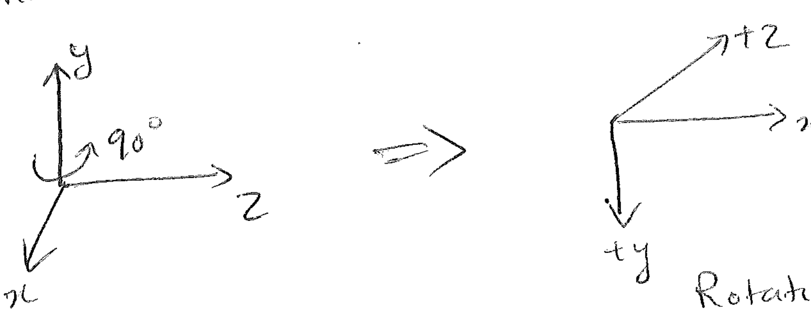
Note: The transformation we are making is like a rotation about the $\hat{y}$ axis by 90°

Thus we expect

$$\hat{S}_z \longrightarrow \hat{S}_x \qquad \text{In the new representation}$$
$$\hat{S}_x \longrightarrow \hat{S}_z$$

Surprise is that $\hat{S}_y \longrightarrow -\hat{S}_y$

This is because the transformation is really a rotation together with inversion about y-axis to preserve the right hand rule



Rotation about y axis by 90° and $y \to -y$

(d) $\hat{S}_x = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\hat{S}_y = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\hat{S}_z = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\hat{S}_x \hat{S}_y = \frac{1}{4}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $\hat{S}_y \hat{S}_x = \frac{1}{4}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

$\hat{S}_y \hat{S}_z = \frac{1}{4}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, $\hat{S}_z \hat{S}_y = \frac{1}{4}\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$

$\hat{S}_x \hat{S}_z = \frac{1}{4}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\hat{S}_z \hat{S}_x = \frac{1}{4}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$[\hat{S}_x, \hat{S}_y] = \frac{i}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i\,\hat{S}_z$

$[\hat{S}_y, \hat{S}_z] = \frac{i}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = i\,\hat{S}_x$

$[\hat{S}_x, \hat{S}_z] = \frac{-i}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -i\,\hat{S}_y$

So $[\hat{S}_i, \hat{S}_j] = i\,\epsilon_{ijk}\hat{S}_k$ in $\hbar$ units.

(e) $\Delta S_x^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2$

$S_x^2 = \frac{1}{4}\mathbb{1}$, $\Delta S_x^2 = \frac{1}{4}\langle \uparrow_z | \mathbb{1} | \uparrow_z \rangle - (\langle \uparrow_z | \hat{S}_x | \uparrow_z \rangle)^2$

$= \frac{1}{4} - 0 = \frac{1}{4}$

$\Delta S_y^2 = \frac{1}{4}\langle \uparrow_z | \mathbb{1} | \uparrow_z \rangle - (\langle \uparrow_z | \hat{S}_y | \uparrow_z \rangle)^2$

$= \frac{1}{4}$

$\Delta S_x = \Delta S_y = 1/2$

now $\langle S_z \rangle = \frac{1}{2}$ $\qquad [\hat{S}_x, \hat{S}_y] = i \hat{S}_z$

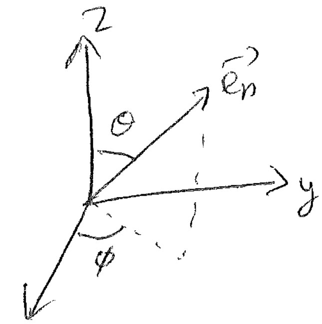
So $\Delta \hat{S}_x \Delta \hat{S}_y \geq \frac{1}{2} \langle \hat{S}_z \rangle$

$$\frac{1}{4} \geq \frac{1}{4} \quad \square$$

It Satisfies uncertainity Principle.

Problem 3: Measurements on a two-State System

Consider direction in 3D space



$$\vec{e}_n = \cos\theta\, \vec{e}_z + \sin\theta\, (\cos\phi\, \vec{e}_x + \sin\phi\, \vec{e}_y)$$

we define the Spin operator along the direction $\vec{e}_n$

$$\hat{S}_n \equiv \hat{\vec{S}} \cdot \vec{e}_n = \sin\theta\, (\cos\phi\, \hat{S}_x + \sin\phi\, \hat{S}_y) + \cos\theta\, \hat{S}_z$$

$$\hat{S}_x = \frac{\hbar}{2} (|\uparrow_z\rangle\langle \downarrow_z| + |\downarrow_z\rangle\langle \uparrow_z|) \equiv \frac{\hbar}{2} \sigma_x$$

$$\hat{S}_y = -\frac{i\hbar}{2} (|\uparrow_z\rangle\langle \downarrow_z| - |\downarrow_z\rangle\langle \uparrow_z|) = \frac{\hbar}{2} \sigma_y$$

$$\hat{S}_z = \frac{\hbar}{2} (|\uparrow_z\rangle\langle \uparrow_z| - |\downarrow_z\rangle\langle \downarrow_z|) = \frac{\hbar}{2} \sigma_z$$

Recall $\sigma_x |\uparrow_z\rangle = |\downarrow_z\rangle \qquad \sigma_y |\uparrow_z\rangle = +i |\downarrow_z\rangle \qquad \sigma_z |\uparrow_z\rangle = |\uparrow_z\rangle$

$\sigma_x |\downarrow_z\rangle = |\uparrow_z\rangle \qquad \sigma_y |\downarrow_z\rangle = -i |\uparrow_z\rangle \qquad \sigma_z |\downarrow_z\rangle = -|\downarrow_z\rangle$

Now define $|\uparrow_n\rangle = \cos\theta/2\, |\uparrow_z\rangle + e^{i\phi} \sin(\theta/2)\, |\downarrow_z\rangle$ as the state with $+\hbar/2$ Spin along $\vec{e}_n$

(a) We seek to show that

$$\hat{S}_n |\uparrow_n\rangle = \frac{\hbar}{2} |\uparrow_n\rangle$$

Proof :

$$\hat{S}_n |\uparrow_n\rangle = \cos\frac{\theta}{2}\, \hat{S}_n |\uparrow_z\rangle + e^{i\phi} \sin\theta/2\, \hat{S}_n |\downarrow_z\rangle$$

$$= \cos\theta/2 \left[\sin\theta(\cos\phi\, \hat{S}_x |\uparrow_z\rangle + \sin\phi\, \hat{S}_y |\uparrow_z\rangle)) + \cos\theta\, \hat{S}_z |\uparrow_z\rangle\right]$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$+ e^{i\phi} \sin\frac{\theta}{2} \left[\qquad |\downarrow_z\rangle + \qquad |\downarrow_z\rangle) + \qquad |\downarrow_z\rangle \right]$$

$$= \frac{\hbar}{2} \cos\frac{\theta}{2} \left[\sin\theta\, (\cos\phi + i\sin\phi)\, |\downarrow_z\rangle + \cos\theta\, |\uparrow_z\rangle\right]$$

$$+ \frac{\hbar}{2} e^{i\phi} \sin\frac{\theta}{2} \left[\sin\theta\, (\cos\phi - i\sin\phi)\, |\uparrow_z\rangle - \cos\theta\, |\downarrow_z\rangle\right]$$

$$= \frac{\hbar}{2} \left(\cos\theta/2\, \cos\theta + \sin\theta/2\, \sin\theta\right) |\uparrow_z\rangle$$

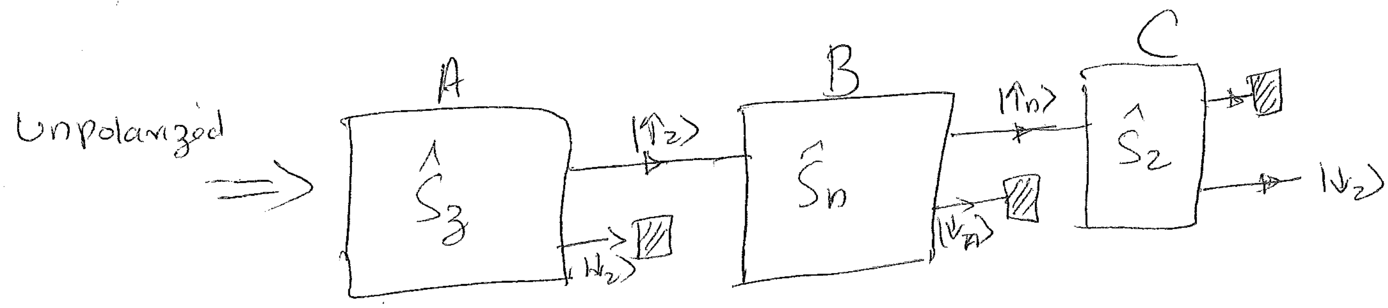
$$+ \hbar/2 \left(\cos\theta/2\, \sin\theta - \sin\theta/2\, \cos\theta\right) e^{i\phi} |\downarrow_z\rangle$$

$$= \frac{\hbar}{2} \left[\cos(\theta - \theta/2)\, |\uparrow_z\rangle + e^{i\phi} \sin(\theta - \theta/2)\, |\downarrow_z\rangle\right]$$

$$= \hbar/2 \left[\cos\theta/2\, |\uparrow_z\rangle + e^{i\phi} \sin\theta/2\, |\downarrow_z\rangle\right]$$

$$\Rightarrow \boxed{\hat{S}_n |\uparrow_n\rangle = \frac{\hbar}{2} |\uparrow_n\rangle}$$ whew!

Now consider following Stern-Gerlach type measurement



(b)

Probability of finding $|\downarrow_z\rangle$ after apparatus C

after atom passes through A atom is in $|\uparrow_z\rangle$ state.

$$P^C(|\downarrow_z\rangle) = P^C(\downarrow_z|\uparrow_n)\, P^B(\uparrow_n|\uparrow_z)$$

$$= |\langle\downarrow_z|\uparrow_n\rangle|^2\, |\langle\uparrow_n|\uparrow_z\rangle|^2$$

Probability that C finds $|\downarrow_z\rangle$ given pure state $|\uparrow_n\rangle$

Probability that B finds $|\uparrow_n\rangle$ given pure state $|\uparrow_z\rangle$

(we have renormalized the result of A)

from part (a) $|\langle\downarrow_z|\uparrow_n\rangle|^2 = \sin^2\theta/2$

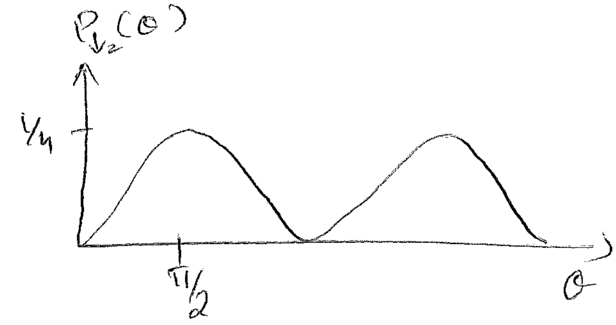
$|\langle\uparrow_n|\uparrow_z\rangle|^2 = \cos^2\theta/2$

$$\Rightarrow \quad = \sin^2\theta/2\, \cos^2\theta/2 = (\sin\theta/2\, \cos\theta/2)^2$$

$$\boxed{P^C(\downarrow_z) = \frac{1}{4}\sin^2\theta}$$

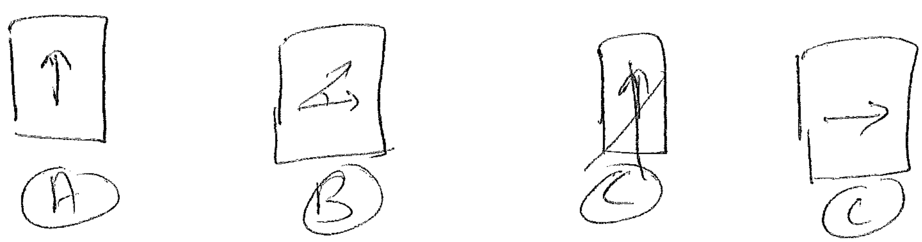
(C) How should we orient the middle apparatus to maximise output

Sketch



Orient $\vec{e}_n$ in the xy plane

This problem is analogous to the optics problem where by we have 3 polarizers



without polarizer (B), (A) & (C) are orthogonal and nothing passes (C). However by inserting the middle polarizer and maximizing the passage of intensity we can transmit $1/4$ of the intensity which passes (A)