Physics 522: Quantum Mechanics I
Problem Set # Solutions

Problem 1: 2D isotropic SHO, encore

(a) $$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{1}{2} m\omega^2 (\hat{x}^2 + \hat{y}^2)$$

Define the characteristic length $l_c = \sqrt{\frac{\hbar}{m\omega}}$, momentum $p_c = \frac{\hbar}{l_c} = \sqrt{\hbar m\omega}$

Define dimensionless position & momentum operators

$$\hat{X} \equiv \frac{\hat{x}}{l_c}, \quad \hat{Y} = \frac{\hat{y}}{l_c}, \quad \hat{P}_x \equiv \frac{\hat{p}_x}{p_c}, \quad \hat{P}_y \equiv \frac{\hat{p}_y}{p_c}$$

$$\Rightarrow \hat{H} = \hbar\omega \left(\frac{\hat{X}^2 + \hat{Y}^2}{2} + \frac{\hat{P}_x^2 + \hat{P}_y^2}{2} \right)$$

Define complex amplitude (anihilation operator) for motion along x and along y

$$\hat{a}_x = \frac{1}{\sqrt{2}} (\hat{X} + i\hat{P}_x), \qquad \hat{a}_y \equiv \left(\frac{\hat{Y} + i\hat{P}_y}{\sqrt{2}} \right)$$

$$\Rightarrow \hat{H} = \hbar\omega(\hat{a}_x^\dagger \hat{a}_x + \tfrac{1}{2}) + \hbar\omega(\hat{a}_y^\dagger \hat{a}_y + \tfrac{1}{2})$$

$$\Rightarrow \boxed{\hat{H} = \hbar\omega(\hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y + 1)}$$

We define the "total number operator"

$$\hat{N} = \hat{N}_x + \hat{N}_y = \hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y$$

$$\Rightarrow \boxed{\hat{H} = \hbar\omega(\hat{N} + 1)}$$

(b.) Give $\hat{H}$ is separable in x + y oscillations

$$\hat{H} = \hat{H}_x \otimes \hat{\mathbb{1}}_y + \hat{\mathbb{1}}_x \otimes \hat{H}_y$$

$\Rightarrow$ Energy eigenvectors are product states of eignstates of $\hat{H}_x$ and $\hat{H}_y$

$$\Rightarrow E_{n_x, n_y} = |n_x\rangle \otimes |n_y\rangle$$

where $|n_i\rangle = \frac{\hat{a}_i^{+}}{\sqrt{n_i!}} |0_i\rangle$ $\quad i = x$ or y

$$\boxed{\hat{H} |n_x\rangle \otimes |n_y\rangle = \hbar\omega(\underbrace{n_x + n_y}_{\equiv n \,=\, 0, 1, 2 \ldots} + 1)}$$

The energy $E_n = \hbar\omega(n + 1)$ with degeneracy.

- For example the first excited state is doubly degenerate: $|1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle$
- The second excited state is triply degenerate: $|2\rangle \otimes |0\rangle$, $|0\rangle \otimes |2\rangle$, and $|1\rangle \otimes |1\rangle$

Generally, the degeneracy is the number of ways of dividing n identical balls into 2 boxes

$$\boxed{g_n = \binom{n+1}{n} = n + 1}$$

(b)[2] Let $\hat{a}_\pm^\dagger = \dfrac{\hat{a}_x^\dagger \pm i\,\hat{a}_y^\dagger}{\sqrt{2}}$

The z-component of angular momentum

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \hbar(\hat{X}\hat{P}_y - \hat{Y}\hat{P}_x)$$

$\hat{X} = \dfrac{\hat{a}_x + \hat{a}_x^\dagger}{\sqrt{2}}$, $\hat{P}_X = \dfrac{\hat{a}_x - \hat{a}_x^\dagger}{i\sqrt{2}}$, and similarly for y

$$\Rightarrow \hat{L}_z = \frac{\hbar}{2i}\left[(\hat{a}_x + \hat{a}_x^\dagger)(\hat{a}_y - \hat{a}_y^\dagger) - (\hat{a}_y + \hat{a}_y^\dagger)(\hat{a}_x - \hat{a}_x^\dagger)\right]$$

$$= \frac{\hbar}{2i}\Big[\hat{a}_x^\dagger\hat{a}_y - \hat{a}_x\hat{a}_y^\dagger + \hat{a}_x\hat{a}_y - \hat{a}_x^\dagger\hat{a}_y^\dagger - \hat{a}_y^\dagger\hat{a}_x + \hat{a}_y\hat{a}_x^\dagger - \hat{a}_y\hat{a}_x + \hat{a}_y^\dagger\hat{a}_x^\dagger\Big]$$

$$\Rightarrow \hat{L}_z = \frac{\hbar}{i}\left[\hat{a}_x^\dagger\hat{a}_y - \hat{a}_y^\dagger\hat{a}_x\right] \quad \left(\text{since } x \text{ and } y \text{ modes commute}\right)$$

Now $\hat{a}_x^\dagger = \dfrac{\hat{a}_+^\dagger + \hat{a}_-^\dagger}{\sqrt{2}}$ $\quad \hat{a}_y^\dagger = \dfrac{\hat{a}_+^\dagger - \hat{a}_-^\dagger}{i\sqrt{2}}$

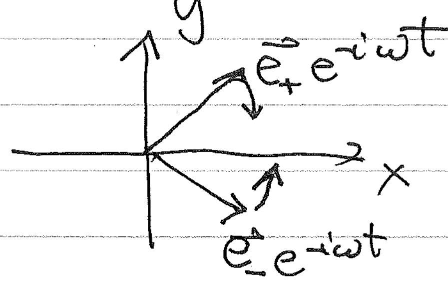
Through similar algebra, with + ~~and~~ − modes commuting

$$\boxed{\begin{aligned}\hat{L}_z &= \hbar(\hat{a}_+^\dagger\hat{a}_+ - \hat{a}_-^\dagger\hat{a}_-) \\ &= \hbar(\hat{N}_+ - \hat{N}_-)\end{aligned}}$$

Physical interpretation:

Recall $\vec{e}_\pm = \frac{\vec{e}_x \pm i\vec{e}_y}{\sqrt{2}}$ represent a vector rotating in a ~~...~~ circle either clockwise or counter-clockwise. More precisely:

$$\vec{r}(t) = \mathrm{Re}\left[\vec{e}_\pm\, r_0 e^{-i\omega t}\right] = \mathrm{Re}\left[r_0 e^{-i\omega t}\right]\vec{e}_x$$
$$\pm \mathrm{Re}\left[i r_0 e^{-i\omega t}\right]\vec{e}_y$$
$$= r_0\left[\cos\omega t\, \vec{e}_x \pm \sin\omega t\, \vec{e}_y\right]$$



That is, the superposition of two orthogonal linear oscillations that are $\pm\frac{\pi}{2}$ (90°) out of phase represent circular motion with $\vec{e}_+$ ($\vec{e}_-$) being positive (negative).

Thus $\hat{a}_\pm^\dagger$ created one quantum of oscillation for rotation with $\pm$ helicity: $\hat{N}_\pm \equiv \hat{a}_\pm^\dagger \hat{a}_\pm$ counts the number of quanta with $\pm$ helicity. Each quantum has $\pm\hbar$ units of angular momentum

$$\Rightarrow \hat{L}_z = \hbar N_+ + (-\hbar)\hat{N}_-$$

(c) The Hamiltonian is also separable in $\pm$ helcity modes

$$\hat{N} = \hat{N}_x + \hat{N}_y = \hat{N}_+ + \hat{N}_- = \hat{a}_+^\dagger \hat{a}_+ + \hat{a}_-^\dagger \hat{a}_-$$

$$\Rightarrow \hat{H} = \underbrace{\hbar\omega(\hat{a}_+^\dagger \hat{a}_+ + \tfrac{1}{2})}_{\hat{H}_+ \otimes \hat{\mathbb{1}}_-} + \underbrace{\hbar\omega(\hat{a}_-^\dagger \hat{a}_- + \tfrac{1}{2})}_{\hat{\mathbb{1}}_+ \otimes \hat{H}_-}$$

Energy eigenstates are product states of eigenstates of $\hat{H}_\pm$ $\quad |n_+\rangle \otimes |n_-\rangle$

$$\hat{H}\, |n_+\rangle \otimes |n_-\rangle = \hbar\omega(\underbrace{n_+ + n_-}_{=n} + 1)$$

Note $\hat{L}_z = \hbar(\hat{N}_+ - \hat{N}_-)$ commutes with $\hat{H}$

$\Rightarrow$ they share common eigenvectors. These are $|n_+\rangle \otimes |n_-\rangle$

$$\boxed{\hat{L}_z\, |n_+\rangle \otimes |n_-\rangle = \hbar(\underbrace{n_+ - n_-}_{\equiv m = 0, \pm 1, \pm 2, \ldots}) = m\hbar}$$

(d) We define eigenvectors of the complete set of mutually commuting operators $\{\hat{N}, \hat{L}_z\}$

$$|n, m\rangle \equiv |n_+ = \tfrac{n+m}{2}\rangle \otimes |n_- = \tfrac{n-m}{2}\rangle$$

$$\hat{N}\,|n, m\rangle = n\,|n, m\rangle, \qquad \hat{L}_z\,|n, m\rangle = \hbar m\,|n, m\rangle$$

We ~~se~~ see, ~~for~~ a given n, m ranges between $-n$ and $+n$ in units of 2 $\Rightarrow$ spin $n/2$

$\Rightarrow$ $\boxed{\text{Degeneracy} \quad n+1}$