Physics 522, Spring 2016
Problem Set #3
Due: Thrursday Feb. 11, 2016 @ 5PM

Problem 1: The Isotropic 2D Harmonic Oscillator Encore (25 Points)

Consider a particle moving in a 2D isotropic harmonic potential
$\hat{V}(x,y) = \frac{1}{2}m\omega^2(\hat{x}^2 + \hat{y}^2)$.

(a) Show that the Hamiltonian commutes with $\hat{L}_z$. Please interpret.

(b) Defining the usual polar coordinates (ρ,ϕ) via $x = \rho\cos\phi$, $y = \rho\sin\phi$, show that the Hamiltonian can be written as

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_\rho^2 + \frac{\hat{L}_z^2}{\rho^2}\right) + \frac{1}{2}m\omega^2\hat{\rho}^2,$$

where $\hat{p}_\rho^2 = -\hbar^2\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)$ is the radial momentum squared and $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$.

(c) Separating coordinates , writing the energy eigenstates $\psi_{n,m}(\rho,\phi) = R_{n,m}(\rho)\Phi_m(\phi)$, with $\Phi_m(\phi)$ the usual eigenstates of $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$, find the "radial equation, for $R_{n,m}(\rho)$.

(d) In class we solved this problem, separating in Cartesian coordinates
$\Psi_{n_x,n_y}(x,y) = u_{n_x}(x)u_{n_y}(y)$
where $u_{n_x}(x)$ and $u_{n_y}(y)$ are the usual energy eigenfunctions in 1D. The energy eigenvalues are
$E_n = \hbar\omega(n+1)$ with degeneracy $n+1$. It must be possible to expand these eigenfunction in terms of the eigenfunction in polar coordinates. Express the five eigenfunctions $\Psi_{0,0}(x,y)$, $\Psi_{1,0}(x,y)$, $\Psi_{0,1}(x,y)$, $\Psi_{1,1}(x,y)$, $\Psi_{2,0}(x,y)$, $\Psi_{0,2}(x,y)$ in this basis. That is write

$$\Psi_{n_x,n_y}(x,y) = \sum_{m=?}^{?} c_{n,m}\,\psi_{n,m}(\rho,\rho), \quad n = n_x + n_y,$$

and thus find the eigenstates $\psi_{n,m}(\rho,\phi)$ for n=0,1, and 2.

(e) Show that $R_{n,m}(\rho)$ satisfy the radial equation you found in (c)

(f) Finally, given the state $\Psi_{1,1}(x,y)$, what are the probabilities of finding the system with angular momentum eigenvalues $m = 2,1,0,-1,-2$ respectively?

Problem 2: From Cohen-Tannoudji vol. I, Exercise 14, page 347. (20 points).

Consider a physical system whose state space, which is three dimensional, spanned by the orthonormal basis formed by the three kets $|u_1\rangle, |u_2\rangle, |u_3\rangle$. In this basis, the Hamiltonian operator of the system and the two observables $\hat{A}$ and $\hat{B}$ have the representations:

$$\hat{H} \doteq \hbar\omega_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \quad \hat{A} \doteq a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \quad \hat{B} \doteq b \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where ω_0, a and b are positive real constants.

The state at time t=0 is: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$.

(a) The energy of the system is measured (projectively) at t=0. What values can be found and with what probabilities? What is the mean value and rms uncertainty in energy?
(b) If one measures observable A at t=0 (projectively), what values can be found and with what probabilities? What is the state vector immediately after the measurement?
(c) Find $|\psi(t)\rangle$.
(d) Calculate $\langle\hat{A}\rangle(t)$ and $\langle\hat{B}\rangle(t)$, for time t>0. What comments can be made?
(e) What results can be obtained if observable A is measured (projectively) at time t? Repeat for B. Interpret.

Problem 3: Simultaneous eigenstates of $\hat{L}^2$ and $\hat{L}_x$ (20 Points)

Though we typically choose the "quantization-axis" as the z-axis, there's nothing that forces this choice. We instead might consider the simultaneous eigenstates of $\hat{L}^2$ and $\hat{L}_x$ by,

$$\hat{L}^2|l,m_x\rangle = \hbar l(l+1)|l,m_x\rangle \text{ and } \hat{L}_x|l,m_x\rangle = \hbar m_x|l,m_x\rangle, \; -l \le m_x \le l \text{ in integer steps.}$$

(a) Using solely the angular momentum algebra, show that

$$\langle \hat{l}_y \rangle_{l,m_x} = \langle \hat{l}_z \rangle_{l,m_x} = 0$$ (*Hint*: Consider commutators)

(b) Argue through symmetry that $\langle \Delta l_y^2 \rangle_{l,m_x} = \langle \Delta l_z^2 \rangle_{l,m_x} = \frac{l(l+1) - m_x^2}{2}$.

(c) Suppose the state of the system is described by the eigenstate $|l=1, m_x=0\rangle$.

Write the normalized wave function for this state in position-space.

(Hint: The easiest way the approach this is to consider the spherical basis appropriate to rotations about the ***x-axis***, as a cyclic permutation of the basis for rotations about z).

(d) Given the state in part (c), what are the possible values that can be measured for $\hat{l}_z$, and with what relative probabilities?

(e) Repeat part (d) for the state $|l=1, m_x=1\rangle$.