Physics 522, Spring 2016
Problem Set #3
Solutions

Problem 1: The Isotropic 2D Harmonic Oscillator Encore

(a) Given a particle in a 2D isotropic oscillator

$$\hat{V}(\hat{x},\hat{y}) = \frac{1}{2} m\omega^2 (\hat{x}^2 + \hat{y}^2)$$

The Hamiltonian separates in Cartesian Coordinates.

$$\hat{H} = \underbrace{\frac{\hat{P}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}_0^2}_{\hat{H}_x} + \underbrace{\frac{\hat{P}_y^2}{2m} + \frac{1}{2} m\omega^2 \hat{y}^2}_{\hat{H}_y}$$

$\Rightarrow$ Eigenstates : $|\Psi_{n_x,n_y}\rangle = |n_x\rangle \otimes |n_y\rangle$

1D oscillator states

Eigen values $E_{n_x,n_y} = \hbar\omega(n_x+n_y+1) = \hbar\omega(n+1)$

$n = n_x + n_y$ Degeneracy $g_n = n+1$

The degeneracy arises because of rotational Symmetry about Z-axis generated by $\hat{L}_z$ (ie. $\hat{L}_z$ is conserved.

Proof $[\hat{H},\hat{L}_z] = \frac{1}{2m}[\hat{P}_x^2+\hat{P}_y^2, \hat{L}_z] + \frac{1}{2}m\omega^2[\hat{x}^2+\hat{y}^2, \hat{L}_z]$

We showed in Prev. HW $[\hat{L}_i, \hat{P}_j] = i\epsilon_{ijk}\hat{P}_k$

$[\hat{L}_i, \hat{x}_j] = i\epsilon_{ijk}\hat{x}_k$

$$[L_z, \hat{p}_x^2] = \hat{p}_x[\hat{L}_z, \hat{p}_x] + [\hat{L}_z, \hat{p}_x]\hat{p}_x = 2i\hat{p}_x\hat{p}_y$$

$$[L_z, \hat{p}_y^2] = \hat{p}_y[\hat{L}_z, \hat{p}_y] + [\hat{L}_z, \hat{p}_y]\hat{p}_y = -2i\hat{p}_x\hat{p}_y$$

Similarly $[\hat{L}_z, \hat{x}^2] = 2i\hat{x}\hat{y}$

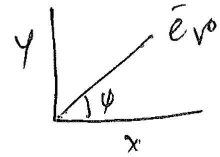
$[\hat{L}_z, \hat{y}^2] = -2i\hat{x}\hat{y}$

$\therefore\ [\hat{H}, \hat{L}_z] = 0$

(b) Defining the usual Polar coordinate

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$



$\rho = \sqrt{x^2+y^2}$

$\varphi = \tan^{-1}(y/x)$

In Polar representation, $\hat{p}_x^2 + \hat{p}_y^2 = -\hbar^2\nabla^2 = -\hbar^2\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)$

We recall $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar\frac{\partial}{\partial\varphi}$ $\Rightarrow$ $\hat{p}_x^2 + \hat{p}_y^2 = \hat{p}_\rho^2 + \frac{\hat{L}_z^2}{2m}$

$$\boxed{\therefore\ \hat{H} = \frac{p_\rho^2}{2m} + \frac{L_z^2}{2m} + \frac{1}{2}m\omega^2\rho^2}$$

(c) We seek Simultaneous eigenfunctions of $\{\hat{H}, \hat{L}_z\}$

we make the ansatz $|\psi_{n,m_\ell}\rangle = |R_{n,m_\ell}\rangle \otimes |\Phi_{m_\ell}\rangle$

where $L_z|\Phi_{m_\ell}\rangle = \hbar m_\ell|\Phi_{m_\ell}\rangle$ eigenstate of $\hat{L}_z$

in position representation

$$-i\frac{\partial}{\partial\varphi}\Phi_{m_\ell}(\varphi) = m_\ell\Phi_{m_\ell}(\varphi)$$

$\Rightarrow\ \Phi_{m_\ell}(\varphi) = \frac{1}{\sqrt{2\pi}}e^{-im_\ell\varphi}$ $\quad m_\ell$ - Integer for Single valued function

Normalized $\int_0^{2\pi}|\varphi_{m_\ell}(\varphi)|^2\,d\varphi = 1$

Now plugging in our ansatz into the Schrödinger eqn:

$$\hat{H}|\psi_{n,m_\ell}\rangle = E_{n,m_\ell}|\psi\rangle$$

Projecting out one m_ℓ value

$$\Rightarrow \left(\underbrace{\frac{\hat{P}_\rho^2}{2m}}_{\text{mess here}} + \frac{\hbar^2 m_\ell^2}{2m\rho^2} + \frac{1}{2}m\omega^2\rho^2\right)|R_{n,m_\ell}\rangle = E_{n,m_\ell}|R_{n,m_\ell}\rangle$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R_{n,m_\ell}}{\partial\rho}\right) + \left(\frac{\hbar m_\ell^2}{2m\rho^2} + \frac{1}{2}m\omega^2\rho^2\right)R_{n,m_\ell} = E_{n,m_\ell}R$$

Define the dimensionless variables in the usual way.

Characteric radius: $r_c = \sqrt{\frac{\hbar}{m\omega}}$, Characteristic energy $E_c = \hbar\omega$

Let $\bar{\rho} \equiv \rho/r_c$ $\quad \bar{E}_{n,m_\ell} \equiv \frac{E_{n,m_\ell}}{\hbar\omega} = n+1$ (from part (a))

$$\Rightarrow \boxed{\left[\bar{\rho}^2 R''_{n,m_\ell} + \bar{\rho}R'_{n,m_\ell} - \bar{\rho}^4 R_{n,m_\ell} + 2(n+1)\bar{\rho}^2 R_{n,m_\ell} - m_\ell^2 R_{n,m_\ell}\right] = 0}$$

Radial equation: Here $R'_{n,m_\ell} \equiv \frac{d}{d\bar{\rho}}R_{n,m_\ell}(\rho)$

(d) We can try to solve the equation directly.

Instead, here I will construct the radial wavefunctions from the familiar separation in Cartesian coordinates

$$\Psi_{n_x,n_y}(X,Y) = U_{n_x}(X)\,U_{n_y}(Y) \qquad \left(\begin{array}{c}\text{Dimensionless}\\ \text{units}\end{array}\right)$$

$$\text{where } U_{n_x}(X) = A_n \underbrace{H_{n_x}(X)}_{\text{Hermite polynomial}} e^{-X^2/2}, \quad A_n = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}$$

$$\therefore\ \Psi_{0,0}(X,Y) = \frac{1}{\sqrt{\pi}}\, e^{-\frac{X^2+Y^2}{2}} = \frac{1}{\sqrt{\pi}}\, e^{-\bar\rho^2/2}$$ (Unique Ground state)

$$\Rightarrow \Psi_{0,0}(X,Y) = \psi_{n=0,\,m_\ell=0}(\bar\rho,\phi) = R_{n=0,\,m_\ell=0}(\bar\rho)\ \underbrace{\Phi_0(\phi)}_{\frac{1}{\sqrt{2\pi}}}$$

$$\Rightarrow \boxed{R_{0,0}(\bar\rho) = \sqrt{2}\, e^{-\bar\rho^2/2}}$$

For n_x or $n_y > 0$ we have degeneracies. However we can always expand one basis in terms of the other.

$$\Psi_{n_x,n_y}(X,Y) = \Psi_{n_x,n_y}(X=\bar\rho\cos\phi,\ Y=\bar\rho\sin\phi)$$

$$= \sum_{m_\ell} C_{m_\ell,\,n=n_x+n_y}\ \psi_{n,m_\ell}(\bar\rho,\phi)$$

Or $$U_{n_x}(\bar\rho\cos\phi)\, U_{n_y}(\bar\rho\sin\phi) = \sum_{m_\ell} \underbrace{C_{m_\ell,\,n=n_x+n_y}}_{\text{expansion coefficients}}\ R_{n,m_\ell}(\bar\rho)\, \frac{e^{-im_\ell\phi}}{\sqrt{2\pi}}$$

Examples

$$n_x=1,\ n_y=0 \Rightarrow \Psi_{n_x,n_y}(X,Y) = \left(\frac{2X}{\sqrt{2\sqrt{\pi}}}\, e^{-X^2/2}\right)\left(\frac{e^{-Y^2/2}}{\sqrt[4]{\pi}}\right)$$

$$\Rightarrow \Psi_{n_x,n_y}(X,Y) = \frac{\sqrt{2}}{\sqrt{\pi}}\,\bar\rho\cos\phi\ e^{-\bar\rho^2/2} = \sqrt{\frac{1}{2\pi}}\ \bar\rho\, e^{-\bar\rho^2/2}\left(e^{+i\phi} + e^{-i\phi}\right)$$

$$\therefore\ \boxed{\Psi_{n_x=1,\,n_y=0}(X,Y) = \bar\rho\, e^{-\bar\rho^2/2}\,\Phi_1(\phi) + \bar\rho\, e^{-\bar\rho^2/2}\,\Phi_{-1}(\phi)}$$

$$\Rightarrow \boxed{R_{n=1,\,m_\ell=1}(\bar\rho) = \sqrt{2}\,\bar\rho\, e^{-\bar\rho^2/2}}$$

$$\boxed{R_{n=1,\,m_\ell=-1}(\bar\rho) = \sqrt{2}\,\bar\rho\, e^{-\bar\rho^2/2}}$$

The factor $\sqrt{2}$ ensures normalization

$$\int_0^\infty d\rho\ \rho\, |R_{n m_\ell}|^2 = 1$$

Similarly

• $\underline{n_x=0,\ n_y=1}$: $\Psi_{n_x=0,n_y=1}(\bar{X},\bar{Y}) = \sqrt{\frac{2}{\pi}}\,\bar{\rho}\sin\phi\, e^{-\bar{\rho}^2/2} = \bar{\rho}\, e^{-\frac{\bar{\rho}^2}{2}} \left(\frac{e^{i\phi} - e^{-i\phi}}{i\sqrt{2\pi}}\right)$

$$\Rightarrow \boxed{\Psi_{n_x=0,n_y=1}(\bar{X},\bar{Y}) = \frac{i}{\sqrt{2}}\left(R_{n=0,m_\ell=1}(\bar{\rho})\,\Phi_1(\phi) - R_{n=0,m_\ell=-1}(\bar{\rho})\,\Phi_{-1}(\phi)\right)}$$

Normalized

$$\boxed{R_{n=0,m_\ell=\pm 1} = \sqrt{2}\,\bar{\rho}\, e^{-\bar{\rho}^2/2}}$$

Turning to the 3 degenerate $n=2$ states

• $\underline{n_x=2,\ n_y=0}$: $\Psi_{n_x=2,n_y=0}(\bar{X},\bar{Y}) = \left(\frac{4\bar{X}^2-2}{\sqrt{8\sqrt{\pi}}}\, e^{-\bar{X}^2/2}\right)\left(\frac{e^{-\bar{Y}^2/2}}{\sqrt[4]{\pi}}\right)$

$$\Rightarrow \Psi_{n_x=2,n_y=0}(\bar{X},\bar{Y}) = \left(\frac{2\bar{\rho}^2\cos^2\phi - 1}{\sqrt{2\pi}}\right) e^{-\bar{\rho}^2/2} = \left(\frac{\bar{\rho}^2-1}{\sqrt{2\pi}} + \bar{\rho}^2\frac{\cos 2\phi}{\sqrt{2\pi}}\right) e^{-\bar{\rho}^2/2}$$

$$\Rightarrow \Psi_{n_x=2,n_y=0}(\bar{X},\bar{Y}) = (\bar{\rho}^2-1)\, e^{-\bar{\rho}^2/2}\,\frac{1}{\sqrt{2\pi}} + \frac{1}{2}\rho$$

$$= (\bar{\rho}^2-1)e^{-\bar{\rho}^2/2}\frac{1}{\sqrt{2\pi}} + \frac{1}{2}\bar{\rho}^2 e^{-\bar{\rho}^2/2}\frac{e^{-i2\phi}}{\sqrt{2\pi}} + \frac{1}{2}e^{-\bar{\rho}^2/2}\frac{e^{+i2\phi}}{\sqrt{2\pi}}$$

$$\Rightarrow \boxed{\Psi_{n_x=2,n_y=0}(\bar{X},\bar{Y}) = \frac{1}{\sqrt{2}}R_{20}(\bar{\rho})\,\Phi_0(\phi) + \frac{1}{2}R_{2,2}(\bar{\rho})\,\Phi_2(\phi) + \frac{1}{2}R_{2,-2}(\bar{\rho})\,\Phi_{-2}(\phi)}$$

Normalized

$$\boxed{\begin{aligned} R_{n=2,m_\ell=0} &= \sqrt{2}\,(\bar{\rho}^2-1)\, e^{-\bar{\rho}^2/2} \\ R_{n=2,m_\ell=\pm 2} &= \bar{\rho}^2\, e^{-\bar{\rho}^2/2} \end{aligned}}$$

• $\underline{n_x=0,\ n_y=2}$: Same procedure $\Psi_{n_x=0,n_y=2} = \left(\frac{2\bar{\rho}^2\sin^2\phi - 1}{\sqrt{2\pi}}\right) e^{-\bar{\rho}^2/2}$

$$\Rightarrow \boxed{\Psi_{n_x=0,n_y=2}(\bar{X},\bar{Y}) = \frac{1}{\sqrt{2}}R_{20}(\bar{\rho})\,\Phi_0(\phi) - \frac{1}{2}R_{2,2}(\bar{\rho})\,\Phi_2(\phi) - \frac{1}{2}R_{2,-2}(\bar{\rho})\,\Phi_{-2}(\phi)}$$

$\underline{n_x = 1,\ n_y = 1}$ $\quad \Psi_{n_x=1, n_y=1}(\bar{X}, \bar{Y}) = \dfrac{(2\bar{X})(2\bar{Y})}{2\sqrt{\pi}} e^{-\frac{\bar{X}^2+\bar{Y}^2}{2}}$

$$\Rightarrow \Psi_{n_x=1, n_y=1} = \frac{\bar{\rho}^2 \cos\phi \sin\phi}{2\sqrt{\pi}} e^{-\bar{\rho}^2/2} = \bar{\rho}^2 \frac{e^{-\bar{\rho}^2/2}}{\sqrt{\pi}} \sin 2\phi$$

$$\Rightarrow \Psi_{n_x=1, n_y=1} = \frac{i}{\sqrt{2}} \bar{\rho}^2 e^{-\bar{\rho}^2/2} \Phi_2(\phi) - \frac{i}{\sqrt{2}} \bar{\rho}^2 e^{-\bar{\rho}^2/2} \Phi_{-2}(\phi)$$

$$\boxed{\Psi_{n_x=1, n_y=1} = \frac{i}{\sqrt{2}} R_{2,2}(\bar{\rho})\ \Phi_2(\phi) - \frac{i}{\sqrt{2}} R_{2,2}(\bar{\rho})\ \Phi_{-2}(\phi)}$$

<u>Summary</u>

$$\Psi_{n_x, n_y}(\bar{X}, \bar{Y}) = \sum_{\substack{m_\ell = -n \\ \text{steps of } 2}}^{n} C_{n, m_\ell} R_{n, m_\ell}(\bar{\rho})\ \Phi_{m_\ell}(\phi) \quad , \ n = n_x + n_y$$

Angular wavefunction: $\Phi_{m_\ell}(\phi) = \dfrac{1}{\sqrt{2\pi}} e^{-i m_\ell \phi}$, Normalized $\int_0^{2\pi} d\phi\, |\Phi_m|^2 = 1$

Radial wave function

Normalized $\int_0^\infty d\bar{\rho}\ \bar{\rho}\ |R_{n,m_\ell}|^2 = 1$

$$\left[\begin{array}{c} R_{0,0} = \sqrt{2}\, e^{-\bar{\rho}^2/2} \ ,\ R_{1,\pm 1} = \sqrt{2}\, \bar{\rho}\, e^{-\bar{\rho}^2/2} \ ,\ R_{2,\pm 2} = \bar{\rho}^2 e^{-\bar{\rho}^2/2} \\ R_{2,0} = \sqrt{2}\, (\bar{\rho}^2 - 1)\, e^{-\bar{\rho}^2/2} \end{array} \right]$$

Expansion coefficients

$n_x = 1,\ n_y = 0$: $\quad C_{1,1} = C_{1,-1} = \frac{1}{\sqrt{2}} \qquad C_{1,0} = 0 \leftarrow$ Parity in the x-y plane

$n_x = 0,\ n_y = 1 \quad C_{1,1} = -C_{1,-1} = \frac{i}{\sqrt{2}}$

$n_x = 2,\ n_y = 0 \quad C_{2,2} = C_{2,-2} = \frac{1}{2},\ C_{2,0} = \frac{1}{\sqrt{2}}$

$n_x = 0\ \ n_y = 2 \quad C_{2,2} = C_{2,-2} = -\frac{1}{2},\ C_{2,0} = \frac{1}{\sqrt{2}}$

$n_x = 1,\ n_y = 1 \quad C_{2,2} = -C_{2,-2} = \frac{i}{\sqrt{2}}$

(e) Show that R_{n,m_ℓ} satisfy the radial eqn

<u>Example:</u> $R_{1,\pm 1} = \rho e^{-\rho^2/2}$ (excluding normalization)

Radial eqn: $\rho^2 R''_{1,1} + \rho R'_{1,1} + (-\rho^4 + 2(2)\rho^2 - 1) R_{1,1} \stackrel{?}{=} 0$

$R'_{1,1} = (1-\rho^2) e^{-\rho^2/2} \qquad R''_{1,1} = (-3\rho + \rho^3) e^{-\rho^2/2}$

$\Rightarrow \rho^2(-3\rho + \rho^3) + \rho(1-\rho^2) + (4\rho^2 - 1 - \rho^4)\rho \stackrel{?}{=} 0$

$\Rightarrow \rho^5 - 3\rho^3 + \rho - \rho^3 + 4\rho^3 - \rho - \rho^5 \stackrel{?}{=} 0$

$0 = 0$ ✓

<u>Example</u>: $R_{2,0} = (\rho^2 - 1) e^{-\rho^2/2}$

$\rho^2 R''_{2,0} + \rho R'_{2,0} + (-\rho^4 + 2(3)\rho^2 - 0) R_{2,0} \stackrel{?}{=} 0$

$R'_{2,0} = (-\rho^3 - 3\rho) e^{-\rho^2/2} \qquad R''_{2,0} = (\rho^4 - 6\rho + 3) e^{-\rho^2/2}$

$\Rightarrow \rho^2(\rho^4 - 6\rho^2 + 3) + \rho(-\rho^3 + 3\rho) + (-\rho^4 + 6\rho^2)(\rho^2 - 1) \stackrel{?}{=} 0$

$\cancel{\rho^6} - \cancel{6\rho^4} + \cancel{3\rho^2} - \cancel{\rho^4} + \cancel{3\rho^2} - \cancel{\rho^6} + \cancel{\rho^4} + \cancel{6\rho^4} - \cancel{6\rho^2} \stackrel{?}{=} 0$

$0 = 0$ ✓

<u>etc.</u>

(f) Given the normalized wave functions, we have the probability amplitudes in the expansions, summarized in part (d)

eg. $\Psi_{n_x=1, n_y=1} = \frac{i}{\sqrt{2}} R_{22}(\tilde{\rho})\, \Phi_2(\phi) - \frac{i}{\sqrt{2}} R_{2,-2}(\tilde{\rho})\, \Phi_{-2}(\phi)$

$\Rightarrow$ Probability of finding $m_\ell = 2$

= Probability of finding $m_\ell = -2$ = $\frac{1}{2}$

0 Probability to find $m_\ell = 0, \pm 1$

Problem 2:

$$\hat{H} = \hbar\omega_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \hat{A} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\hat{B} = b \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle \quad \{|u_i\rangle \text{ orthonormal}\}$$

(a) Measure energy of system at $t=0$

Can find energy eigenvalues.

$\boxed{E_1 = \hbar\omega_0}$ with Probability $P_1 = |\langle u_1|\psi(0)\rangle|^2 = \frac{1}{2}$

$\boxed{E_2 = 2\hbar\omega_0 = E_3}$ with Probability $P_2 = |\langle u_2|\psi(0)\rangle|^2 + |\langle u_3|\psi(0)\rangle|^2$

$= \frac{1}{4} + \frac{1}{4} = 1/2$

(Since $|u_2\rangle$ & $|u_3\rangle$ are degenerate)

(b) ~~Instead we measure $\hat{A}$. We must find eigenvalues of $\hat{A}$~~

Mean value $\langle \hat{H}\rangle = \left(\cancel{\frac{1}{\sqrt{2}}\langle u_1| + \frac{1}{2}\langle u_2|} + \frac{1}{2}\langle u_3|\right.$

$= \frac{1}{2}E_1 + \frac{1}{2}E_2 = \frac{\hbar\omega_0}{2} + \hbar\omega_0 = \frac{3}{2}\hbar\omega_0$

$H|\psi(0)\rangle = \hbar\omega_0\left(\frac{1}{\sqrt{2}}|u_1\rangle + |u_2\rangle + |u_3\rangle\right)$

$\langle\psi(0)|H^2|\psi(0)\rangle = \hbar^2\omega_0^2\left(\frac{1}{2} + 1 + 1\right) = \frac{5}{2}\hbar^2\omega_0^2$

$\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2} = \sqrt{\frac{5}{2} - \frac{9}{4}}\,\hbar\omega_0 = \frac{\hbar\omega_0}{2}$

(b) Instead of $\hat{H}$ we measure $\hat{A}$. We must find its eigenvalues.

$\hat{A}$ is in block-diagonal form

$$\hat{A} = a\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$$

clearly $|u_1\rangle$ is an eigenvector with eigenvalue a

We must diagonalize $\hat{A}$ in the subspace spanned by $|u_2\rangle$ & $|u_3\rangle$

$$\hat{A} \doteq a\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = a\,\hat{\sigma}_x$$

where $\hat{\sigma}_x$ is the Pauli spin matrix in the basis where $|\uparrow\rangle = |u_2\rangle$ & $|\downarrow\rangle = |u_3\rangle$. We have already solved this.

eigen system is

$$\lambda = a \quad |a\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(|u_2\rangle + |u_3\rangle\right)$$

$$\lambda = -a \quad |-a\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(|u_2\rangle - |u_3\rangle\right)$$

So overall eigen system of $\hat{A}$ is

$$\lambda = a \quad |a^{(1)}\rangle = |u_1\rangle, \quad |a^{(2)}\rangle = \frac{1}{\sqrt{2}}\left(|u_2\rangle + |u_3\rangle\right)$$

$$\lambda = -a \quad |-a\rangle = \frac{1}{\sqrt{2}}\left(|u_2\rangle - |u_3\rangle\right)$$

Suppose the state $|\psi(0)\rangle$ enters the apparatus which "measures" $\hat{A}$. We can find.

- Eigenvalue $-a$ with probality $P_{-a} = |\langle -a|\psi(0)\rangle|^2$

$$\Rightarrow P_{-a} = \left|\left(\frac{\langle u_2| - \langle u_3|}{\sqrt{2}}\right)|\psi(0)\rangle\right|^2$$

$$= 0 \quad (\text{No component of } |-a\rangle \text{ in } |\psi(0)\rangle$$

(If such a component had existed the state immediately after the measurement would have been $|-a\rangle$)

- Eigenvalue $+a$: This is a degenerate subspace.

The probability of finding eigenvalue a

$$P_a = \langle\psi(0)|\hat{\mathbb{P}}_a|\psi(0)\rangle, \text{ where } \hat{\mathbb{P}}_a = |a^{(1)}\rangle\langle a^{(1)}| + |a^{(2)}\rangle\langle a$$

$$= |\langle a^{(1)}|\psi(0)\rangle|^2 + |\langle a^{(2)}|\psi(0)\rangle|^2$$

$$= |\langle u_1|\psi(0)\rangle|^2 + \left|\left(\frac{\langle u_2| + \langle u_3|}{\sqrt{2}}\right)|\psi(0)\rangle\right|^2$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \quad (\text{as expected since } P_a + P_{-a} = 1)$$

Immediately after the measurement, according to Von Neuman projection postulate

$$|\psi_{out}\rangle = \frac{\hat{\mathbb{P}}_a|\psi(0)\rangle}{\|\hat{\mathbb{P}}_a|\psi(0)\rangle\|} = |\psi(0)\rangle$$

No change since $|\psi(0)\rangle$ is an eigenstate of $\hat{A}$

(c) $|\psi(t)\rangle = U(t)|\psi(0)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$

$$= \frac{1}{\sqrt{2}} e^{-iE_1 t/\hbar}|u_1\rangle + \frac{e^{-iE_2 t/\hbar}}{2}(|u_2\rangle + |u_3\rangle)$$

$$\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_0 t}|u_1\rangle + \frac{1}{2}e^{-i2\omega_0 t}(|u_2\rangle + |u_3\rangle)}$$

(d) $$\langle \hat{A}(t)\rangle = \begin{bmatrix} \frac{e^{+i\omega_0 t}}{\sqrt{2}} & \frac{e^{+i2\omega_0 t}}{2} & \frac{e^{+i2\omega_0 t}}{2} \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{bmatrix} \begin{bmatrix} e^{-i\omega_0 t}/\sqrt{2} \\ e^{-2i\omega_0 t}/2 \\ e^{-2i\omega_0 t}/2 \end{bmatrix}$$

[using representation $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$]

$$\Rightarrow \boxed{\langle \hat{A}(t)\rangle = \frac{a}{2} + \frac{a}{4} + \frac{a}{4} = a}$$

As expected since $|\psi(0)\rangle$ is eigenstate of

$$\langle \hat{B}(t)\rangle = \begin{bmatrix} \frac{e^{i\omega_0 t}}{\sqrt{2}} & \frac{e^{2i\omega_0 t}}{2} & \frac{e^{2i\omega_0 t}}{2} \end{bmatrix} \begin{bmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & b \end{bmatrix} \begin{bmatrix} e^{-i\omega_0 t}/\sqrt{2} \\ e^{-2i\omega_0 t}/2 \\ e^{-2i\omega_0 t}/2 \end{bmatrix}$$

$$= \frac{b}{\sqrt{2}}\left(\frac{e^{-i\omega_0 t}}{2} + \frac{e^{i\omega_0 t}}{2}\right) + \frac{b}{4}$$

$$\boxed{\langle \hat{B}(t)\rangle = \frac{b}{\sqrt{2}}\cos\omega_0 t + \frac{b}{4}}$$

Note that $\langle \hat{A}\rangle$ is <u>independent of time</u> while $\langle \hat{B}\rangle$ is not. This ~~followse~~ follows directly from the fact that

$$[\hat{H}, \hat{A}] = 0 \quad \text{while} \quad [\hat{H}, \hat{B}] \neq 0$$

(e) Suppose we measure $\hat{A}$ at time t

Since $[\hat{A},\hat{A}]=0$ the probabilities of measuring different values of $\hat{A}$ do not change with time, thus, the results are the same as for part (b)

Suppose we measure $\hat{B}$ at time t.
The eigenvectors/values can easily be found

Eigenvalue: b (doubly degenerate)

$$|b^{(1)}\rangle = \frac{1}{\sqrt{2}}\left(|u_1\rangle + |u_2\rangle\right) \qquad |b^{(2)}\rangle = |u_3\rangle$$

Probability of finding "b" at time t

$$P_b(t) = |\langle b^{(1)}|\psi(t)\rangle|^2 + |\langle b^{(2)}|\psi(t)\rangle|^2$$

$$= \left|\frac{e^{-i\omega_0 t}}{\sqrt{2}\sqrt{2}} + \frac{e^{-2i\omega_0 t}}{2\sqrt{2}}\right|^2 + \left|\frac{e^{-2i\omega_0 t}}{2}\right|^2$$

$$\boxed{P_b(t) = \frac{\sqrt{2}}{4}\cos(\omega t) + \frac{5}{8}}$$

Eigenvalue: $-b$ (Nondegenerate)

$$P_{-b}(t) = |\langle -b|\psi(t)\rangle|^2 = \left|\left(\frac{\langle u_1| - \langle u_2|}{\sqrt{2}}\right)|\psi(t)\rangle\right|^2 = \left|\frac{e^{-i\omega_0 t}}{2} - \frac{e^{-}}{2\sqrt{}}\right.$$

$$\boxed{P_{-b}(t) = -\frac{\sqrt{2}}{4}\cos(\omega t) + \frac{3}{8}}$$

Note: $P_b(t) + P_{-b}(t) = 1$ $\langle\hat{B}\rangle = bP_b(t) - bP_{-b}(t) = \frac{b}{\sqrt{2}}\cos\omega t + \frac{b}{4}$

Problem 3 : Simultaneous eigen states of $\hat{L}^2$ & $\hat{L}_x$

$$\hat{L}^2 |l, m_x\rangle = \hbar^2 l(l+1) |l, m_x\rangle \;, \quad \hat{L}_x |l, m_x\rangle = \hbar m_x |l, m_x\rangle$$

$$-l \leq m \leq l$$

(a) Using Angular momentum Algebra

$$\hat{L}_y = (i\hbar)^{-1} [\hat{L}_z, \hat{L}_x] \;, \quad \hat{L}_z = (i\hbar)^{-1} [\hat{L}_x, \hat{L}_y]$$

$$\therefore \langle \hat{L}_y \rangle = \frac{1}{i\hbar} \langle l, m_x | [\hat{L}_z, \hat{L}_x] | l, m_x\rangle = \frac{1}{i\hbar} \langle l, m_x | \hat{L}_z \hat{L}_x - \hat{L}_x \hat{L}_z | l, m_x\rangle$$

$$= \frac{1}{i\hbar} \left(m_x \langle l, m_x | \hat{L}_z | l, m_x\rangle - m_x \langle l, m_x | \hat{L}_z | l, m_x\rangle \right) = 0$$

Similarly $\langle \hat{L}_z \rangle = \frac{1}{i\hbar} \left(m_x \langle \hat{l}_y \rangle - m_x \langle \hat{l}_y \rangle \right) = 0$

(b) $$\langle l, m_x | \hat{l}_y^2 | l, m_x\rangle = - \langle l, m_x | [\hat{l}_z, \hat{l}_x]^2 | l, m_x\rangle$$

$$= - \langle (\hat{l}_z \hat{l}_x - \hat{l}_x \hat{l}_z)(\hat{l}_z \hat{l}_x - \hat{l}_x \hat{l}_z) \rangle$$

$$= - \langle \hat{l}_z \hat{l}_x \hat{l}_z \hat{l}_x \rangle + \langle \hat{l}_z \hat{l}_x^2 \hat{l}_z \rangle$$

$$+ \langle l_x l_z^2 l_x \rangle - \langle l_x l_z l_x l_z \rangle$$

$$= - m_x \langle \hat{l}_z \hat{l}_x \hat{l}_z \rangle + \langle \hat{l}_z \hat{l}_x^2 \hat{l}_z \rangle$$

$$+ m_x^2 \langle \hat{l}_z^2 \rangle - m_x \langle \hat{l}_z \hat{l}_x \hat{l}_z \rangle$$

$$= -2m_x \langle \hat{l}_z \hat{l}_x \hat{l}_z \rangle + \langle \hat{l}_z l_x^2 \hat{l}_z \rangle + m_x^2 \langle \hat{l}_z^2 \rangle$$

Now using the commutator.

$$\langle \hat{l}_z \hat{l}_x \hat{l}_z \rangle = \langle l_z^2 l_x \rangle + \langle l_z [l_x, l_z] \rangle = m_x \langle l_z^2 \rangle - i \langle l_z l_y \rangle$$

$$\langle l_z l_x^2 l_z \rangle = \langle l_z l_x l_z l_x \rangle + \langle l_z l_x [l_x, l_z] \rangle$$

$$= m_x \langle l_z l_x l_z \rangle - i \langle l_z l_x l_y \rangle$$

$= m_x^2 \langle l_z^2 \rangle - i m_x \langle l_z l_y \rangle - i(m_x \langle l_z l_y \rangle) + \langle l_z^2 \rangle$

$= (m_x^2 + 1)\langle l_z^2 \rangle - 2 i m_x \langle l_z l_y \rangle$

$\langle l_y^2 \rangle = -2m_x (m_x \langle l_z^2 \rangle - i \langle l_z l_y \rangle) + (m_x^2+1) \langle l_z^2 \rangle$
$\quad - 2 i m_x \langle l_z l_y \rangle + m_x^2 \langle l_z^2 \rangle$

$= \langle l_z^2 \rangle$ Whew! All, that work for something that is obvious from symmetry

Now, $\hat{l}^2 = \hat{l}_x^2 + \hat{l}_y^2 + \hat{l}_z^2$

$\langle l^2 \rangle = \langle l_x \rangle^2 + 2 \langle l_y^2 \rangle$

$l(l+1) = m_x^2 + 2\langle l_y^2 \rangle$

$\langle l_y \rangle^2 = \langle l_z^2 \rangle = \dfrac{l(l+1) - m_x^2}{2}$

Since $\langle \hat{l}_y \rangle = \langle \hat{l}_z \rangle = 0$

$$\boxed{\langle \Delta l_y^2 \rangle = \langle \Delta l_z^2 \rangle = \frac{l(l+1) - m_x^2}{2}}$$

Result expected from cyclic Permutation on usual eigenstate of $\hat{l}_z$.

(c) Consider now the state $|l=1, m_x=0\rangle$

What is the normalized wave function?

We know in the case of eigenstates of $\hat{l}_z$

$$\langle \vec{e}_r | l=1, m_z=0\rangle = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

Normalization + phase-convention

Now with the x-direction as the "quantization-axis" we make the cyclic permutation

old → new

$z \rightarrow x = r\sin\theta\cos\phi$

$x \rightarrow y = r\sin\theta\sin\phi$

$y \rightarrow z = r\cos\theta$

$$\Rightarrow \boxed{\langle \vec{e}_r | l=1, m_x=0\rangle = \sqrt{\frac{3}{4\pi}}\,\frac{x}{r} = \sqrt{\frac{3}{4\pi}}\sin\theta\cos\phi}$$

Check: We know that a polynomial of 1^{st} power in x is an eigenstate of $\hat{l}^2$ with eigenvalue 1. Lets check

$$\hat{l}_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$\Rightarrow \hat{l}_x \frac{x}{r} = 0 \quad ✓$$

(d) ~~Problem~~ Decomposing $|l=1, m_x=0\rangle$ onto $|l, m_z\rangle$ eigenstates

We have: $\langle \vec{e}_r | l=1, m_z=\pm 1\rangle = \mp\sqrt{\frac{3}{8\pi}}\left(\frac{x\pm iy}{r}\right)$

$$\Rightarrow \frac{x}{r} = -\sqrt{\frac{8\pi}{3}}\left(\frac{\langle \vec{e}_r | l=1, m_z=1\rangle - \langle \vec{e}_r | l=1, m_z=-1\rangle}{2}\right)$$

$$\Rightarrow \langle \vec{e}_r | l=1, m_x=0\rangle = \sqrt{\frac{3}{4\pi}}\,\frac{x}{r} = -\frac{1}{\sqrt{2}}\langle \vec{e}_r | l=1, m_z=1\rangle + \frac{1}{\sqrt{2}}\langle \vec{e}_r | l=1, m_z=-1\rangle$$

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$$\Rightarrow \boxed{|l=1, m_x=0\rangle = \frac{-|l=1, m_z=1\rangle + |l=1, m_z=-1\rangle}{\sqrt{2}}}$$

$\Rightarrow$ Probability of measuring $\hat{l}_z$ with $m_z = +1$: $1/2$
" " " " " " $= -1$: $1/2$

(e) Repeating with $|l=1, m_x=1\rangle$

We have $\langle \vec{e}_r | l=1, m_z=1\rangle = -\sqrt{\frac{3}{8\pi}}\left(\frac{x+iy}{r}\right)$

Cyclic permutation $\Rightarrow$ $\boxed{\langle \vec{e}_r | l=1, m_x=1\rangle = -\sqrt{\frac{3}{8\pi}}\left(\frac{y+iz}{r}\right)}$

$$\frac{y}{r} = -\sqrt{\frac{8\pi}{3}}\left(\frac{\langle \vec{e}_r | l=1, m_z=1\rangle + \langle \vec{e}_r | l=1, m_z=-1\rangle}{2i}\right)$$

$$\frac{z}{r} = \sqrt{\frac{4\pi}{3}}\ \langle \vec{e}_r | l=1, m_z=0\rangle$$

$$\Rightarrow \boxed{|l=1, m_x=1\rangle = \frac{i}{2}|l=1, m_z=1\rangle + \frac{i}{2}|l=1, m_z=-1\rangle - \frac{i}{\sqrt{2}}|l=1, m_z=0\rangle}$$

$\Rightarrow$ Probability of finding eigenvalues of $\hat{l}_z$:

$m_z = +1$	:	$1/4$
$m_z = 0$	:	$1/2$
$m_z = -1$	:	$1/4$