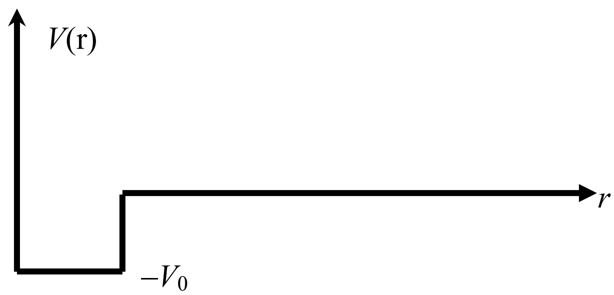
Physics 522, Spring 2016

Problem Set #5

Due: Thursday Feb. 25, 2016 @ 5PM

Problem 3: The Finite Spherical Well (20 points)

Consider a spherically symmetric potential, $V(r)=\begin{cases}-V_0 & 0<r<a \\ 0 & r>a\end{cases}$. Along the radial coordinate, due to the boundary condition at r=0, this is just the half-finite well we studied in Problem Sets 5 and 6.



(a) For E<0, the solutions to the T.I.S.E. are bound states. Let $E=-E_b$. Making the ansatz for the stationary state wave functions $\psi_{E,l,m}(r,\theta,\phi)=R_{E,l}(r)Y_l^m(\theta,\phi)$, show that the radial function must have the form,

$$R_{E,l}(r)=\begin{cases} A\ j_l(k_1 r) & 0<r<a \\ \\ A\,\dfrac{j_l(k_1 a)}{h_l^{(1)}(\kappa a)}h_l^{(1)}(i\kappa r) & r>a \end{cases},\quad \text{where } k_1=\sqrt{\frac{2m}{\hbar^2}(V_0-E_b)},\ \kappa=\sqrt{\frac{2m}{\hbar^2}E_b}\ .$$

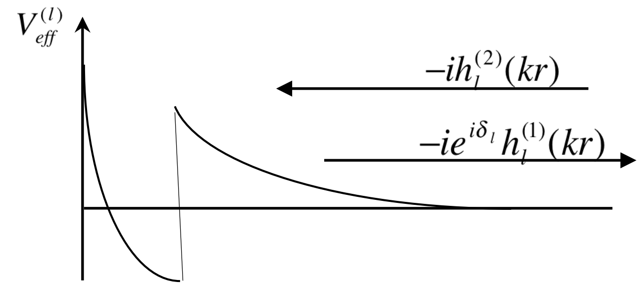
How would you determine A?

(b) Show that the binding energies are determined by the transcendental equation

$$\left(\frac{\frac{d}{dr}(rj_l(k_1 r))}{rj_l(k_1 r)}\right)_{r=a}=\left(\frac{\frac{d}{dr}\left(rh_l^{(1)}(i\kappa r)\right)}{rh_l^{(1)}(i\kappa r)}\right)_{r=a}.$$

Does this reduce to the expected solution of s-states (i.e. $l=0$).

(c) Now consider the unbound states. We seek the scattering phase shift for the asymptotic incoming and outgoing partial waves, as discussed in Lecture.



Show that the phase satisfies the equation

$$\left(\frac{rj_l(qr)}{\frac{d}{dr}\left[rj_l(qr)\right]}\right)_{r=a} = \left(\frac{r\left(\cos(\delta_l/2)\,j_l(kr)-\sin(\delta_l/2)n_l(kr)\right)}{\frac{d}{dr}\left[r\left(\cos(\delta_l/2)\,j\ (ka)-\sin(\delta_l/2)n_l(kr)\right)\right]}\right)_{r=a},$$

$$\text{where } k=\sqrt{\frac{2m}{\hbar^2}E} \text{ and } q=\sqrt{\frac{2m}{\hbar^2}(E+V_0)}.$$

Check that this limits to the expected result for *s*-wave (*l*=0).

Problem 2: The 3D Isotropic Simple Harmonic Oscillator. (20 points)

Consider a particle of mass *m* moving in a three dimensional isotropic SHO, with frequency ω.
(a) Since the problem is separable in Cartesian coordinates, show that the energy eigenvalues are

$E_n = \hbar\omega(n+3/2)$, where $n = 0,1,2,\ldots$ Show that the degeneracy is $g_n = \dfrac{(n+1)(n+2)}{2}$.

(b) The degeneracy is of course stemming from the rotational symmetry of the problem. Let us now seek simultaneous eigenfunctions of $\left\{\hat{H},\hat{L}^2,\hat{L}_z\right\}$ and separate in spherical coordinates, so that the wave function is $\psi_{n_r,l,m}(r,\theta,\phi)=\dfrac{u_{n_r,l}(r)}{r}Y_{l,m}(\theta,\phi)$. Defining the usual dimensionless variables $\bar{r}\equiv r/r_c$, $\varepsilon\equiv E/\hbar\omega$, (where $r_c=\sqrt{\hbar/m\omega}$), write the radial equation of the *reduced* radial wave function in dimensionless units, and show that it must have the form,

$$u_{n_r,l}(\bar{r})=\bar{r}^{\,l+1}e^{-\bar{r}^2/2}F_{n_r,l}(\bar{r}),$$

where $F_{n_r,l}(\bar{r})$ is constant near the origin, and does not blow up faster than $e^{\bar{r}^2}$ for large $\bar{r}$.

(c) Show that in fact, the radial wave functions are,

$$R_{n_r,l}(\bar{r}) = \bar{r}^l e^{-\bar{r}^2/2} \mathsf{L}_{n_r}^{l+1/2}(\bar{r}^2), \text{ (unnormalized)}$$

where $\mathsf{L}_p^q(x)$ are the associated Laguerre polynomials.

$E_{n_r,l,m} = \hbar\omega(2n_r + l + 3/2) = \hbar\omega(n+3/2)$, where the "principal quantum number" is defined by $n = 2n_r + l$. Sketch the first three degenerate energy levels, and label the *s,p,d* states. Show again that the degeneracy of the energy eigenvalues you found are as in part (a).

Problem 3: Hydrogenic atoms and atomic units. (15 points)

Consider the "hydrogenic" atoms - that is bound-states of two oppositely charged particles:

(i) The hydrogen atom: Binding of an electron and proton.
(ii) Heavy ion: Single electron bound to a nucleus of mass *M*, charge *Ze* (say Z=50).
(iii) Muonium: Muon bound to a proton
(iv) Positronium: Bound state of an electron and a positron (anti-electron)

(a) For each, using the charges, *reduced* mass, and the unit $\hbar$, determine the characteristic scales of:

Length, energy, time, momentum, internal electric field, and electric dipole moment.

Please give numerical values as well as the expressions in terms of the fundamental constants.

(b) Now add the speed of light *c* into the mix. Find characteristic velocity in units of *c*, magnetic field, and magnetic moment. Show that for the particular case of hydrogen the characteristic velocity is $v/c = \alpha = \frac{e^2}{\hbar c}(cgs) \approx \frac{1}{137}$, the "fine-structure" constant, and that the Bohr radius, Compton wavelength, and "classical electron radius", differ by powers of α according to,

$$r_{class} = \alpha\lambda_{compton} = \alpha^2 a_0$$

(c) What is the characteristic magnetic field and magnetic dipole moment?