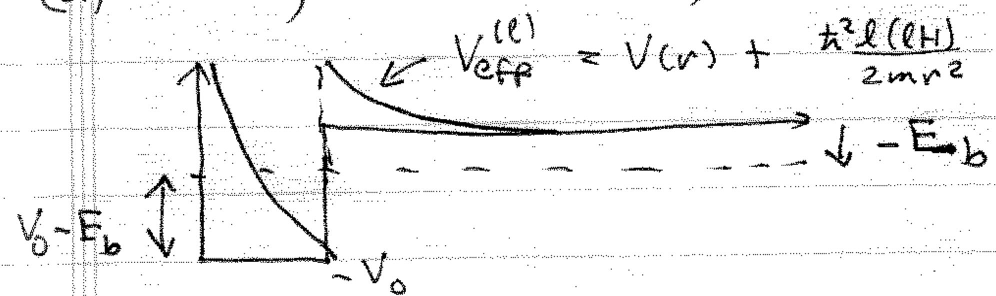
Physics 522: Problem Set #5 Solutions

Problem 1: The Finite Spherical Well

$$V(r) = \begin{cases} -V_0 & 0<r<a \\ 0 & r>a \end{cases}$$

(a) $E<0$, Bound state, let $E = -E_b$.



In each region we have a constant potential where the solution is generally

$$R_\ell(r) = A j_\ell(kr) + B n_\ell(kr)$$

or $$= C h_\ell^{(1)}(kr) + D h_\ell^{(2)}(kr)$$

Where $k = \sqrt{\frac{2m}{\hbar^2}(E - V(r))}$ constant in the region

The coefficents A, B or C, D are set by the boundary conditions and normalization.

- For $0 < r < a$, we must have $R_\ell(r=0)$ finite

$$\Rightarrow R_\ell^{(I)}(r) = A j_\ell(qr) \qquad (B = 0)$$

where $q = \sqrt{\frac{2m}{\hbar^2}(-E - (-V_0))} = \sqrt{\frac{2m}{\hbar^2}(V_0 - E_b)}$

- For $a < r < \infty$ we must have $\lim_{r \to \infty} R_\ell \to 0$

Now $h_\ell^{(1/2)}(kr) = j_\ell(kr) \pm i n_\ell(kr) \propto e^{\pm ikr}$

In region ~~the~~ II $k = \sqrt{\frac{2m E}{\hbar^2}} = \sqrt{\frac{-2mE_b}{\hbar^2}} = i\kappa$

(classically forbidden) Pure imaginary

$$\Rightarrow R_\ell^{(II)}(r) = C h_\ell^{(1)}(i\kappa r) \quad \text{where } \kappa = \sqrt{\frac{2mE_b}{\hbar^2}}$$

Finally using continuity of the wave function

$$R_\ell^{(I)}(a) = R_\ell^{(II)}(a) \Rightarrow C h_\ell^{(1)}(i\kappa a) = A j_\ell(qa)$$

$$\Rightarrow C = A j_\ell(qa) / h_\ell^{(1)}(i\kappa a)$$

$$\Rightarrow \boxed{R_\ell(r) = \begin{cases} A j_\ell(qr) & 0 < r < a \\ A \dfrac{j_\ell(qa)}{h_\ell^{(1)}(i\kappa a)} h_\ell^{(1)}(i\kappa r) & a < r < \infty \end{cases}}$$

A: determined by normalization

(b) We now have two unknowns: A and E_b

- A is set by normalization
- E_b is set by the remaining b.c.

$$\frac{d}{dr} u_\ell^{(I)}(a) = \frac{d}{dr} u_\ell^{(II)}(a)$$

where $u_\ell(r) = r R_\ell(r)$ is the reduced radial wave function

$$\Rightarrow A \left.\frac{d}{dr}\left(r j_\ell(qr)\right)\right|_a = A \frac{j_\ell(qa)}{h_\ell^{(1)}(i\kappa a)} \left.\frac{d}{dr}\left(r h_\ell^{(1)}(i\kappa a)\right)\right|_a$$

$$\Rightarrow \boxed{\frac{\left.\frac{d}{dr}\left(r j_\ell(qr)\right)\right|_a}{j_\ell(qa)} = \frac{\left.\frac{d}{dr}\left(r h_\ell^{(1)}(i\kappa a)\right)\right|_a}{h_\ell^{(1)}(i\kappa a)}}$$

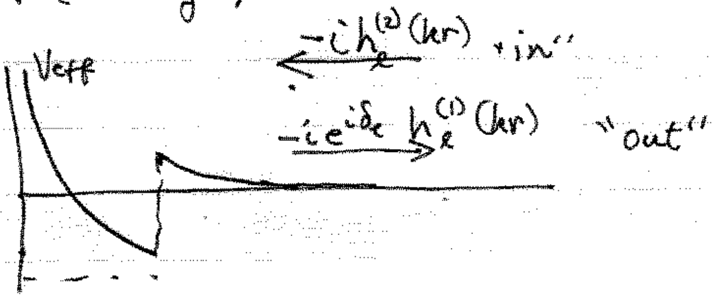
Check: for $\ell = 0$, $V_{eff}^{(\ell=0)} = V(r)$.

Recall $j_0(qr) = \frac{\sin qr}{qr}$ $\qquad h_\ell^{(1)}(i\kappa r) = -\frac{e^{-\kappa r}}{\kappa r}$

$$\Rightarrow \left.\frac{d}{dr}\left(r j_0(qr)\right)\right|_a = \frac{\cos(qa)}{\cancel{a}}, \quad \left.\frac{d}{dr}\left(r h_0^{(1)}(i\kappa r)\right)\right|_{r=a} = \frac{e^{\kappa a}}{q}$$

$$\overset{\ell=0}{\Rightarrow} \Rightarrow q\frac{\cos(qa)}{\sin(qa)} = -\kappa \Rightarrow \boxed{\kappa = -q\cot(qa)}$$

We now consider the unbound energy eigenstates. We seek the scattering phase shift according to the asymptotic conditions



Find δ_ℓ

- For $r>a$ $\quad R_\ell(r) = -i\left(h_\ell^{(2)}(kr) + e^{i\delta_\ell} h_\ell^{(1)}(kr)\right)$

Note $h_\ell^{(1,2)}(kr) = j_\ell(kr) \pm i n_\ell(kr)$

$\Rightarrow r>a \quad R_\ell(r) = -i(1+e^{i\delta_\ell})\, j_\ell(kr) + (-1+e^{i\delta_\ell})\, n_\ell(kr)$

$$R_\ell(r) = -2i\, e^{i\delta_\ell/2}\left[\cos\frac{\delta_\ell}{2}\, j_\ell(kr) - \sin\frac{\delta_\ell}{2}\, n_\ell(kr)\right]$$

- For $0<r<a$ $\quad R_\ell(r) = A\, j_\ell(kr)$ (to remain regular at origin)

Logarithmic derivative:

$$\left(\frac{1}{R_\ell}\frac{dR_\ell}{dr}\right)^{(I)}\Bigg|_{r=a} = \left(\frac{1}{R_\ell}\frac{dR_\ell}{dr}\right)^{II}\Bigg|_{r=a}$$

$$\Rightarrow \boxed{\frac{j_\ell(qa)}{q\left.\frac{dj_\ell}{dr}\right|_{qa}} = \frac{\cos\frac{\delta_\ell}{2}\, j_\ell(ka) - \sin\frac{\delta_\ell}{2}\, n_\ell(ka)}{k\left[\cos\frac{\delta_\ell}{2}\left.\frac{dj_\ell}{dr}\right|_{ka} - \sin\frac{\delta_\ell}{2}\left.\frac{dn_\ell}{dr}\right|_{ka}\right]}}$$

Alternately we can use the reduced equation

$$\boxed{\left.\frac{r\,j_\ell(qr)}{\frac{d}{dr}\left(r\,j_\ell(qr)\right)}\right|_{r=a} = \left.\frac{\left(r\cos\frac{\delta_\ell}{2}\,j_\ell(kr) - r\sin\frac{\delta_\ell}{2}\,n_\ell(kr)\right)}{\frac{d}{dr}\left[r\cos\frac{\delta_\ell}{2}\,j_\ell(kr) - r\sin\frac{\delta_\ell}{2}\,n_\ell(kr)\right]}\right|_{r=a}}$$

Now check: $\ell = 0$ $\quad r\,j_0(qr) = \frac{1}{q}\sin(qr)$

$$r\,n_0(qr) = -\frac{1}{q}\cos qr$$

$$\Rightarrow \frac{\sin qa}{q\cos(qa)} = \frac{\cos\frac{\delta_0}{2}\sin(ka) + \sin\frac{\delta_0}{2}\cos(ka)}{k\left(\cos\frac{\delta_0}{2}\cos(ka) - \sin\frac{\delta_0}{2}\cos(ka)\right)}$$

$$\Rightarrow \frac{1}{q}\tan(qa) = \frac{\sin\left(ka + \frac{\delta_0}{2}\right)}{k\cos\left(ka + \frac{\delta_0}{2}\right)} = \frac{1}{k}\tan\left(ka + \frac{\delta_0}{2}\right)$$

$$\Rightarrow \boxed{\frac{\delta_0}{2} = \tan^{-1}\left(\frac{k}{q}\tan(qa)\right) - ka}$$

Problem 2: The 3D Isotropic SHO

The Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\vec{x}) \qquad \hat{V}(\vec{x}) = \frac{1}{2}m\omega(\hat{x}^2 + \hat{y}^2 + \hat{z}^2) = \frac{1}{2}m\omega\hat{r}^2$$

(a) This problem is separable in Cartesian coordinates. The energy eigenkets are

$$|n_x, n_y, n_z\rangle = |n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle \quad \begin{pmatrix}\text{Product of three} \\ \text{1D oscillators}\end{pmatrix}$$

With energy eigenvalues $E_n = \hbar\omega(n + 3/2)$ where $n = n_x + n_y + n_z$

The degeneracy of the states with energy E_n is as follows

We showed in class that the degeneracy of a 2D isotropic SHO with eigenvalues $E = \hbar\omega(n_x + n_y + 1)$ is $n_x + n_y + 1$

If we fix n_z then $n_x + n_y = n - n_z \rightarrow$ Degeneracy $n - n_z + 1$

Thus as we allow n_z to range over all possible values for a given n (i.e. $0 \le n_z \le n$), the total degeneracy is

$$g_n^{3D} = \sum_{n_z=0}^{n} g_{n-n_z}^{2D} = \sum_{n_z=0}^{n} (n+1-n_z) = (n+1)^2 - \sum_{n_z=0}^{n} n_z$$

$$\left(\text{Aside: } \sum_{i=0}^{N} i = \frac{N(N+1)}{2}\right)$$

$$\Rightarrow \text{Degeneracy } g_n^{3D} = (n+1)^2 - \frac{n(n+1)}{2} = (n+1)\left(n+1-\frac{n}{2}\right)$$

$$\Rightarrow \boxed{g_n^{3D} = \frac{(n+1)(n+2)}{2}}$$

The first few degenerate energy levels with cartesian quantum numbers are shown below

$n=2$ $(g_2=6)$ $|2,0,0\rangle$ $|0,2,0\rangle$ $|0,0,2\rangle$ $|1,1,0\rangle$ $|1,0,1\rangle$ $|0,1,1\rangle$

$n=1$ $(g_1=3)$ $|1,0,0\rangle$ $|0,1,0\rangle$ $|0,0,1\rangle$

$n=0$ $(g_0=1)$ $|0,0,0\rangle$

• the states are denoted by $|n_x, n_y, n_z\rangle$

(b) Because of the rotational symmetry we can seek eigenstates of the complete set of commuting operators

$$\{\hat{H}, \hat{L}^2, \hat{L}_z\}$$

Separating in spherical coordinates with the eigenfunction written the usual way:

$$\psi_{n_r,l,m}(r,\theta,\phi) = R_{n_r,l}(r)\, Y_{l,m}(\theta,\phi)$$

where $R_{n_r l}(r) = \frac{u_{n_r l}(r)}{r}$ ← reduced radial wave function

The radial equation for $u_{n_r l}(r)$ is

$$\left(-\frac{\hbar^2 d^2}{2m\,dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + \frac{1}{2}m\omega^2 r^2\right) u_{n_r l}(r) = E_{n_r l}\, u_{n_r l}(r)$$

Defining the usual dimensionless variables $\bar{r} \equiv \frac{r}{r_c}$, $\varepsilon = \frac{E}{\hbar\omega}$ where the characteristic radius is $r_c = \sqrt{\frac{\hbar}{m\omega}}$

$$\Rightarrow \left(-\frac{1}{2}\frac{d^2}{d\bar{r}^2} + \frac{l(l+1)}{2\bar{r}^2} + \frac{1}{2}\bar{r}^2\right) u(\bar{r}) = \varepsilon\, u(\bar{r})$$

Since $\hat{V}(r)$ does not blow up at the origin, the centripetal barrier dominates there and we expect the asymptotic form:

$$u(\bar{r}) \sim \bar{r}^{l+1} \quad as \quad r \to 0$$

For $r \to \infty$, the potential blows up and dominates over the centripetal barrier and the $\mathcal{E}u$ term

$$\Rightarrow \text{As } r \to \infty \quad \frac{d^2}{d\bar{r}^2} u - \bar{r}^2 u = 0$$

We can solve this dif'eqn by making the substitution $y = \bar{r}^2 \Rightarrow \frac{du}{d\bar{r}} = \frac{dy}{d\bar{r}} \frac{du}{dy} = 2\bar{r} \frac{du}{\partial y}$

$$\Rightarrow \frac{d^2u}{d\bar{r}^2} = 2\frac{du}{dy} + 4y\frac{d^2u}{dy^2} = y\, u(y) \Rightarrow \frac{d^2u}{dy^2} = \frac{-1}{2y}\frac{du}{dy} + \frac{u(y)}{4}$$

We can neglect the second term in asymptotic $y \to \infty$

$$\Rightarrow \frac{d^2u}{dy^2} - \frac{1}{4}u(y) = 0 \Rightarrow u(y) = Ae^{-y/2} + \underbrace{\cancel{Be^{+y/2}}}_{\text{neglect since it diverges at } \infty}$$

$$\Rightarrow \text{As } r \to \infty \quad u(\bar{r}) \sim e^{-\bar{r}^2/2}$$

$\Rightarrow$ We expect the reduced radial wavefunction to have the form

$$\boxed{u_{n,l}(\bar{r}) = \bar{r}^{l+1} e^{-\bar{r}^2/2} F_{n,l}(\bar{r})}$$

where $F_{n,l}(\bar{r})$ is constant near the origin and does not blow up faster than $e^{\bar{r}^2}$

(c) Substitute the Ansatz into the radial equation

$$\left(\frac{d^2}{d\bar{r}^2} - \frac{\ell(\ell+1)}{\bar{r}^2} - \bar{r}^2 + 2\varepsilon\right) u_\ell(\bar{r}) = 0$$

$$u_\ell(r) = g(r)\, F_\ell(r) \quad \text{where} \quad g(r) = \bar{r}^{\ell+1} e^{-\frac{\bar{r}^2}{2}}$$

$$\frac{d^2}{dr^2} u_\ell(\bar{r}) = \frac{d^2 g}{dr^2} F_\ell + g\frac{d^2 F_\ell}{dr} + 2\frac{dg}{dr}\frac{dF_\ell}{d\bar{r}}$$

After some algebra,

$$\frac{d^2}{d\bar{r}^2} u_\ell(\bar{r}) = \left\{ \left(\frac{\ell(\ell+1)}{\bar{r}^2} - (2\ell+3) + \bar{r}^2\right) F_\ell(\bar{r}) + \left(\frac{2\ell+2}{\bar{r}} - 2\bar{r}\right) F_\ell'(\bar{r}) + F_\ell''(\bar{r}) \right\} \bar{r}^{\ell+1} e^{-\frac{\bar{r}^2}{2}}$$

$$\Rightarrow \frac{d^2 F_\ell}{d\bar{r}^2} + 2\left(\frac{\ell+1}{\bar{r}} - \bar{r}\right)\frac{dF_\ell}{d\bar{r}} - (3+2\ell-2\varepsilon)\, F_\ell(\bar{r}) = 0$$

We can put this in the form of the Laguerre equation through a change of variables
Let $x = \bar{r}^2$. Define $F(\bar{r}) = G(x)$ $\Rightarrow$ $\frac{dF(\bar{r})}{d\bar{r}} = 2\bar{r}\frac{dG}{dx}$, $\frac{d^2F}{d\bar{r}^2} = 2\frac{dG}{dx} + 4\bar{r}^2\frac{d^2G}{dx^2}$

$$\left(2\frac{dG_\ell}{dx} + 4\bar{r}^2\frac{d^2G_\ell}{dx^2}\right) + 2\left[\frac{(\ell+1)}{\bar{r}} - \bar{r}\right]\left(2\bar{r}\frac{dG_\ell}{dx}\right) = (3+2\ell-2\varepsilon)\, G_\ell(x)$$

$$\Rightarrow 4x\frac{d^2G_\ell}{dx^2} + 4\left[\ell + \tfrac{3}{2} - x\right]\frac{dG_\ell}{dx} = (3+2\ell-2\varepsilon)\, G_\ell(x)$$

$$\Rightarrow \boxed{x\frac{d^2G_\ell}{dx^2} + \left[1 + \ell + \tfrac{1}{2} - x\right]\frac{dG_\ell}{dx} = \tfrac{1}{2}\left(\tfrac{3}{2} + \ell - \varepsilon\right) G_\ell(x)}$$

This is the Laguerre equation $x\frac{d^2 L^q_{n_r}}{dx^2} + [1+q-x]\frac{dL^q_{n_r}}{dr} = -n_r L^q_{n_r}(x)$
Where $n_r = 0, 1, 2, ..$ determines the order of the polynomial $\Rightarrow F_{n_r,\ell}(\bar{r}) = \mathcal{L}^{\ell+\frac{1}{2}}_{n_r}(\bar{r}^2)$

$$\Rightarrow n_r = -\tfrac{1}{2}\left(\tfrac{3}{2} + \ell - 2\varepsilon\right) \Rightarrow \varepsilon = 2n_r + \ell + \tfrac{3}{2}$$

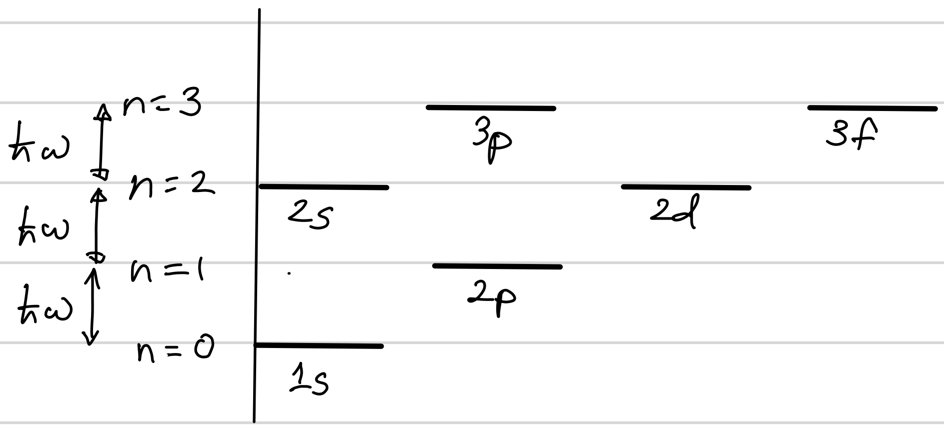
$$\Rightarrow \boxed{E = \hbar\omega\left(2n_r + \ell + \tfrac{3}{2}\right) = \hbar\omega\left(n + \tfrac{3}{2}\right) \quad \text{where} \quad n = 2n_r + \ell}$$

$$\boxed{R_{n_r,\ell}(\bar{r}) = \bar{r}^\ell e^{-\frac{\bar{r}^2}{2}} \mathcal{L}^{\ell+\frac{1}{2}}_{n_r}(\bar{r}^2)}$$

The energy levels are specified by three quantum numbers, n, l, m_l ; the energy eigenvalues depend only on n. The radial quantum number $n_r = \frac{n-l}{2}$. Since n_r is an integer, and $n_r \geq 0$, given a value of n, l ranges over

if n even: $l = 0, 2, \ldots, n$ in steps of 2
if n odd: $l = 1, 3, \ldots, n$ " " " "

We thus have the following energy-level diagram



s states: nondegenerate
p states : 3 fold degenerate
d states : 5 fold degenerate
f states : 7 fold degenerate

Note, the states with given l have parity $(-1)^l$. Thus the n even states are even parity and n odd are odd parity, as expected.

For each l there are $2l+1$ degenerate sublevels. We can thus find the degeneracy g_n

$$n \text{ even} \Rightarrow g_n = \sum_{l=0,2,4}^{n} (2l+1) = \sum_{k=0}^{n/2} (4k+1) = \left(\frac{n}{2}+1\right) + 4\sum_{i=0}^{n/2} k$$

$$= \frac{n+2}{2} + 4\left[\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2}\right] = \frac{n+2+n(n+2)}{2} = \frac{(n+1)(n+2)}{2}$$

$$n \text{ odd} \Rightarrow g_n = \sum_{l=1,3,5\ldots}^{n-1} (2l+1) = \sum_{k=0}^{\frac{n-1}{2}} (4k+3) = \frac{(n+1)(n+2)}{2}$$

Problem 3: Hydrogenic atoms in atomic units

Two oppositely charged particles:

- charge 1 (negative) $q_1 = -Z_1 e$, mass m_1
- Charge 2 (positive) $q_2 = Z_2 e$, mass m_2

Coulomb interaction: $V(r) = \frac{q_1 q_2}{r} = -Z_1 Z_2 \frac{e^2}{r}$

Relative motion Hamiltonian: $\hat{H} = \frac{\hat{p}^2}{2\mu} + V(r)$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad (\text{reduced mass})$$

$\boxed{\text{Characteristic Units}}$: scales determined by $\mu, q_1, q_2, \hbar$

Length: a_c, Momentum: $p_c = \frac{\hbar}{a_c}$, Energy: E_c

Relate: $E_c = \frac{p_c^2}{\mu} = \frac{q_1 q_2}{a_c} = \frac{\hbar^2}{a_c^2 \mu}$

$$\underset{\text{kinetic energy}}{\frac{p_c^2}{\mu}} \qquad \underset{\text{Coulomb energy}}{\frac{q_1 q_2}{a_c}}$$

Length: $\boxed{a_c = \frac{\hbar^2}{\mu q_1 q_2} = \frac{m_e}{\mu Z_1 Z_2}\left(\frac{\hbar^2}{m_e e^2}\right) = \frac{m_e}{\mu Z_1 Z_2}(0.53 \text{Å})}$

Momentum $\boxed{p_c = \frac{\hbar}{a_c} = \left(\frac{\mu Z_1 Z_2}{m_e}\right)\frac{\hbar}{0.53\text{Å}} = \frac{\mu Z_1 Z_2}{m_3} 2\times 10^{-19} \frac{g\,cm}{s}}$

Energy $$E_c = \frac{q_1 q_2}{a_c} = \left(\frac{\mu Z_1^2 Z_2^2}{m_e}\right)\left(\frac{m_e e^4}{\hbar^2}\right) \doteq \left(\frac{\mu Z_1^2 Z_2^2}{m_e}\right) 27.2\,eV$$

Time: $$t_c = \frac{\hbar}{E_c} = \left(\frac{m_e}{\mu Z_1^2 Z_2^2}\right)\frac{\hbar}{E_0} = \left(\frac{m_e}{\mu Z_1^2 Z_2^2}\right) 2.4\times 10^{-17}\,s$$

~10 atto seconds

Internal Electric field @ particle 1 $$\mathcal{E}_c = \frac{q_2}{a_c^2} = \left(\frac{\mu Z_1 Z_2}{m_e}\right)^2 \frac{Z_2 e}{a_0^2} \simeq \left(\frac{\mu^2 Z_1^2 Z_2^3}{m_e^2}\right) 5\times 10^9 \frac{V}{cm}$$

Electric dipole moment $$d_c = \left(\frac{q_1 - q_2}{2}\right) a_c = \left(\frac{Z_1 + Z_2}{2}\right)\left(\frac{m_e}{\mu}\right)\left(\frac{1}{Z_1 Z_2}\right) e a_0$$

Aside $e a_0 = 8.5\times 10^{-30}$ C-m $= 2.54$ Debye

1 debye $= 10^{-10}$ esu $\cdot$ Å

Speed (units of c) $$\frac{v_c}{c} = \frac{a_c}{c t_c} = \frac{a_c}{c \frac{\hbar}{E_c}} = \frac{q_1 q_2}{\hbar c} = (Z_1 Z_2)\frac{e^2}{\hbar c}$$

$$= (Z_1 Z_2)\,\alpha$$ ← fine structure constant

Internal B-field @ particle 1 $$B_c = \frac{v_c}{c} \mathcal{E}_c = (Z_1 Z_2)(\alpha \mathcal{E}_c) = \left(\frac{\mu^2 Z_1^3 Z_2^4}{m_e^2}\right) \boxed{\frac{e}{a_0^2}}\,\alpha$$

~10^5 Gau

Magnetic dipole moment $\mu_c = \frac{\text{Current} \times \text{Area}}{c} = \frac{q_1}{t_c}\frac{a_c^2}{c}$

2× μ_B = Bohr magneton

$$\boxed{\Rightarrow \mu_c = \left(\frac{a_c}{t_c c}\right)(q_1 a_c) = \alpha d_c = \left(\frac{Z_1+Z_2}{2}\right)\left(\frac{m_e}{\mu}\right)\left(\frac{e\hbar}{mc}\right)}$$

Now for each case given:

(i) Hydrogen defines "atomic units":

- $a_c = a_0 = 0.53$ Å Bohr radius
- $E_c = E_0 = 27.2$ eV Hartree
- $t_c = 2.4 \times 10^{-17}$ s
- $\mathcal{E}_c = 5 \times 10^9 \frac{V}{cm} = 1.7 \times 10^7 \frac{\text{statV}}{cm}$
- $B_c = \alpha E_c = 1.2 \times 10^5$ Gauss (Not standard atomic unit)
- $d_c = 2.5 \times 10^{-18}$ cgs $= 2.5$ debye
- $\mu_c = 2\mu_B = \alpha d_c = 1.8$ ergs/Gauss $= 1.8 \times 10^{-24}$ Joule/Tesla

(ii) Heavy ion: $Z_1 = 1$, $Z_2 = 50$, $\mu \simeq m_e$

(iii) Muonium: $m_1 \simeq 200\, m_e$ $m_2 = m_p \simeq 2000 m_p$, $\mu \cong 180 m_e$

(iv) Positronium: $Z_1 = Z_2 = 1$, $m_1 = m_2 = m_e$, $\mu = \frac{m_e}{2}$

Summary table: Characteristic Units in a.u.

	a_c	E_c	t_c	p_c	v_c/c	$\mathcal{E}_c$	B_c	d_c	μ_c
Hydrogen	1	1	1	1	1	1	1	1	1
Sn^{+49}	$\frac{1}{50}$	2500	$\frac{1}{2500}$	50	50	6.25×10^6	6.25×10^6	$\frac{1}{2}$	25
Muonium	$\frac{1}{180}$	180	$\frac{1}{180}$	180	1	32400	32400	$\frac{1}{180}$	$\frac{1}{180}$
Positronium	2	$\frac{1}{2}$	2	$\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{1}{4}$	2	2

From these results we see that
- for heavy elements the electron can be highly relativistic and magnetic effects grow
- Muonium sees huge field because the muon is so close to the nucleus.

Other useful relations

Given constants $e, m_e, \hbar, c$ we have additional scales

- rest energy: $E_{rest} = m_e c^2$

$$\Rightarrow \frac{E_0}{m_e c^2} = \frac{\frac{m_e e^4}{\hbar^2}}{mc^2} = \frac{e^4}{(\hbar c)^2} = \alpha^2$$

$$\Rightarrow E_0 = \alpha^2 mc^2$$

- Compton wavelength: $\lambda\!\!\!^{-}_c = \frac{\hbar}{m_e c} = \alpha a_0$
- Classical electron radius: $r_{class} = \frac{m_e c^2}{e^2} = \frac{\lambda\!\!\!^{-}_c}{\alpha}$