Physics 522: Problem Set #8 Solutions

Problem 1 Addition of spin and orbital angular momentum

An electron has both spin angular momentum, described by operator $\vec{S} = \hbar \vec{s}$ and orbital angular momentum described by operator $\vec{L} = \hbar \vec{l}$.

- We can describe the state in the "uncoupled representation" in terms of simultaneous eigenvectors of $\{\hat{l}^2, \hat{l}_z; \hat{s}^2, \hat{s}_z\}$

$|l m_l\rangle \otimes |s, m_s\rangle$ where $s = 1/2$ for electrons $\Rightarrow m_s = +1/2, -1/2$

Here we consider states with $l = 1 \Rightarrow m_l = 1, 0, -1$

- Alternately, we can use the "coupled representation" in terms of simultaneous eigenvectors of $\{\hat{j}^2, \hat{j}_z, \hat{l}^2, \hat{s}^2\}$

where $\hat{\vec{j}} = \hat{\vec{l}} + \hat{\vec{s}} \Rightarrow \hat{j}_z = \hat{l}_z + \hat{s}_z$,

Eigenvectors: $|j m_j; l s\rangle$

Since $l=1, s=1/2$ is a common eigenvalue in both representations I will denote the short-hand for the eigenvectors

$$\left.\begin{array}{ll} \text{Uncoupled:} & |m_l, m_s\rangle \\ \text{Coupled:} & |j, m_j\rangle \end{array}\right\} \ l, s \text{ understood and common to both}$$

Let us write $\hat{j}^2$ and $\hat{j}_z$ as matrices in the uncoupled basis

We need the relationship: $\hat{j}^2 = \hat{\vec{j}} \cdot \hat{\vec{j}} = \hat{\vec{l}}^2 + \hat{\vec{s}}^2 + 2\hat{l}_z \hat{s}_z + (\hat{l}_+ \hat{s}_- + \hat{l}_- \hat{s}_+)$

$$\Rightarrow \hat{j}_z |m_l, m_s\rangle = (\hat{l}_z + \hat{s}_z)|m_l, m_s\rangle = (m_l + m_s)\, |m_l, m_s\rangle$$

and
$$\hat{j}^2 |m_l, m_s\rangle = \{l(l+1) + s(s+1) + 2m_l m_s\}\, |m_l, m_s\rangle$$
$$+ \sqrt{l(l+1) + m_l(m_l+1)}\ \sqrt{s(s+1) - m_s(m_s-1)}\ |m_l+1, m_s-1\rangle$$
$$+ \sqrt{l(l+1) - m_l(m_l-1)}\ \sqrt{s(s+1) + m_s(m_s+1)}\ |m_l-1, m_s+1\rangle$$

<u>Note:</u> The uncoupled basis vectors $|m_\ell, m_s\rangle$ are <u>already</u> eigenvectors of $\hat{J}_z$, with eigenvalue $m_j = m_\ell + m_s$

Since $m_\ell = 1, 0, -1$ and $m_s = \frac{1}{2}, -\frac{1}{2}$, the possible values of m_j are $m_j = \pm 3/2$ and $\pm 1/2$, with a double degeneracy for $m_j = \pm 1/2$.

$$\begin{cases} m_j = 3/2 & \Rightarrow \quad |m_\ell = 1,\ m_s = 1/2\rangle \\ m_j = 1/2 & \Rightarrow \quad |m_\ell = 0,\ m_s = 1/2\rangle \quad \text{or} \quad |m_\ell = 1,\ m_s = -1/2\rangle \\ m_j = -1/2 & \Rightarrow \quad |m_\ell = 0,\ m_s = -1/2\rangle \quad \text{or} \quad |m_\ell = -1,\ m_s = 1/2\rangle \\ m_j = -3/2 & \Rightarrow \quad |m_\ell = -1,\ m_s = -1/2\rangle \end{cases}$$

We can simplify the calculation by ordering the basis into <u>orthogonal-subspaces</u>. Since the diagonalization of $\hat{J}^2$ cannot mix states with different m_j, we choose the <u>ordered basis</u>

$$|m_\ell, m_s\rangle = \{\underbrace{|1, 1/2\rangle}_{m_j = 3/2};\ \underbrace{|0, \tfrac{1}{2}\rangle, |1, -1/2\rangle}_{m_j = \frac{1}{2}};\ \underbrace{|0, -1/2\rangle, |-1, 1/2\rangle}_{m_j = -1/2};\ \underbrace{|-1, -1/2\rangle}_{m_j = -3/2}\}$$

$$\Rightarrow \quad \hat{J}_z = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix}, \quad \hat{J}^2 = \begin{bmatrix} \frac{15}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{11}{4} & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & \frac{7}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{11}{4} & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{4} \end{bmatrix}$$

Thus in each of the 2D subspaces with $m_j = 1/2$ and $m_j = -1/2$ we must diagonize the same matrix

$$M = \begin{bmatrix} \frac{11}{4} & \sqrt{2} \\ \sqrt{2} & \frac{7}{4} \end{bmatrix}$$

The secular equation for <u>both</u> $m_j = +1/2$ and $m_j = -1/2$ subspaces

$$\det(\lambda \mathbb{1} - M) = (\lambda - \tfrac{11}{4})(\lambda - \tfrac{7}{4}) - 2 = 0$$

$$\Rightarrow \lambda^2 - \tfrac{9}{2}\lambda + \tfrac{45}{16} = 0$$

$$\Rightarrow \lambda = \tfrac{3}{4} \quad \text{or} \quad \lambda = \tfrac{15}{4}$$

Remember, <u>eigenvalue</u> of $\hat{j}^2$ denoted $j(j+1) \Rightarrow j = \frac{1}{2}$ or $\frac{3}{2}$

<u>Eigenvectors</u> <u>$j = \frac{1}{2}$</u>, $\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \frac{a}{b} = -\frac{1}{\sqrt{2}}$

Normalized: $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \end{bmatrix}$ (Remember, this is a representation given an ordered basis)

$j = \frac{3}{2}$: $\begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \frac{a}{b} = \sqrt{2}$: Normalized: $\begin{bmatrix} \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

We thus have the following eigenvectors in the coupled representation:

$$j = 3/2 \begin{cases} |j = 3/2, m_j = +3/2\rangle = |m_\ell = 1, m_s = +1/2\rangle \\ |j = 3/2, m_j = +1/2\rangle = \sqrt{\frac{2}{3}}\,|m_\ell = 0, m_s = 1/2\rangle + \sqrt{\frac{1}{3}}\,|m_\ell = 1, m_s = -1/2\rangle \\ |j = 3/2, m_j = -1/2\rangle = \sqrt{\frac{2}{3}}\,|m_\ell = 0, m_s = -1/2\rangle + \sqrt{\frac{1}{3}}\,|m_\ell = -1, m_s = +1/2\rangle \\ |j = 3/2, m_j = -3/2\rangle = |m_\ell = -1, m_s = -1/2\rangle \end{cases}$$

$$j = 1/2 \begin{cases} |j = 1/2, m_j = +1/2\rangle = \sqrt{\frac{1}{3}}\,|m_\ell = 0, m_s = 1/2\rangle - \sqrt{\frac{2}{3}}\,|m_\ell = 1, m_s = -1/2\rangle \\ |j = 1/2, m_j = -\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}\,|m_\ell = 0, m_s = -1/2\rangle - \sqrt{\frac{2}{3}}\,|m_\ell = -1\ m_s = +1/2\rangle \end{cases}$$

These agree with C-T, Complement A_{X} (vol. II) eqns. 36-a)

Note:

- The possible eigenvalues of j range from $j_{max} = l+s = \frac{3}{2}$ to $j_{min} = |l-s| = \frac{1}{2}$ as expected
- The Clebsch-Gordan coefficents are now given

$$\langle j\, m_j | l\, m_l ; s\, m_s \rangle = \langle l\, m_l ; s\, m_s | j\, m_j \rangle$$

$$\Rightarrow \langle \tfrac{3}{2}\, \tfrac{3}{2} | 1\,1 ; \tfrac{1}{2}\, \tfrac{1}{2} \rangle = \langle \tfrac{3}{2}, -\tfrac{3}{2} | 1\, {-1}, \tfrac{1}{2}\, {-\tfrac{1}{2}} \rangle = 1$$

$$|\langle \tfrac{3}{2}\, \tfrac{1}{2} | 1\,1, \tfrac{1}{2}\, {-\tfrac{1}{2}} \rangle| = |\langle \tfrac{3}{2}, -\tfrac{1}{2} | 1\, {-1}, \tfrac{1}{2}\, \tfrac{1}{2} \rangle| = |\langle \tfrac{1}{2}\, \tfrac{1}{2} | 1\,0 ; \tfrac{1}{2}\, \tfrac{1}{2} \rangle| = |\langle \tfrac{1}{2}\, {-\tfrac{1}{2}} | 1\,0, \tfrac{1}{2}\, {-\tfrac{1}{2}} \rangle| = \sqrt{\tfrac{1}{3}}$$

$$|\langle \tfrac{3}{2}\, \tfrac{1}{2} | 1\,0 ; \tfrac{1}{2}\, \tfrac{1}{2} \rangle| = |\langle \tfrac{3}{2}, -\tfrac{1}{2} | 1\,0, \tfrac{1}{2}\, {-\tfrac{1}{2}} \rangle| = |\langle \tfrac{1}{2}\, \tfrac{1}{2} | 1\,1, \tfrac{1}{2}\, {-\tfrac{1}{2}} \rangle| = |\langle \tfrac{1}{2}\, \tfrac{1}{2} | 1\, {-1}, \tfrac{1}{2}\, \tfrac{1}{2} \rangle| = \sqrt{\tfrac{2}{3}}$$

We cannot assure the sign of CG coeffecent is consistent with our phase convention by this method.

(b) We can easily find the ~~eigenstates~~ matrix elements of $\hat{\vec{l}} \cdot \hat{\vec{s}}$ by noting

$$\hat{\vec{j}}^{\,2} = (\hat{\vec{l}} + \hat{\vec{s}}) \cdot (\hat{\vec{l}} + \hat{\vec{s}}) = \hat{\vec{l}}^{\,2} + \hat{\vec{s}}^{\,2} + 2\, \hat{\vec{l}} \cdot \hat{\vec{s}}$$

$$\Rightarrow \hat{\vec{l}} \cdot \hat{\vec{s}} = \frac{1}{2}\left(\hat{\vec{j}}^{\,2} - \hat{\vec{l}}^{\,2} - \hat{\vec{s}}^{\,2} \right)$$

$$\Rightarrow \langle j' m_j' ; l, s | \hat{\vec{l}} \cdot \hat{\vec{s}} | j\, m_j ; l\, s \rangle = \frac{1}{2}\left(j(j+1) - l(l+1) - s(s+1) \right) \delta_{jj'} \delta_{m_j m_j'}$$

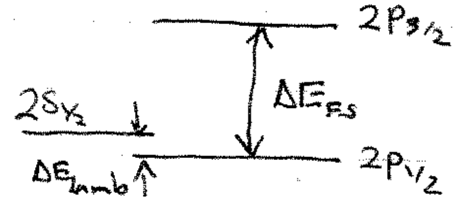
$$\Rightarrow \langle j = \tfrac{1}{2}\, m_j ; l, s | \hat{\vec{l}} \cdot \hat{\vec{s}} | j = \tfrac{1}{2}, m_j, l, s \rangle = -2$$

$$\langle j = \tfrac{3}{2}, m_j ; l\, s | \hat{\vec{l}} \cdot \hat{\vec{s}} | j = \tfrac{3}{2}, m_j, l\, s \rangle = 1$$

$$= 0 \quad \text{therwise}$$

Problem 2 Stark effect ~~with~~ Fine-structure

$n=2$ manifold of Hydrogen (including fine structure)



Stark effect perturbation: $\hat{H}_{int} = +e\hat{z}E_z$ $\begin{pmatrix}\text{quantization} \\ \text{axis along } \vec{E}\end{pmatrix}$

Recall spectroscopic notation: $n l_j \Rightarrow \underset{\uparrow \atop n=2}{2}s_{(1/2)}$ $\leftarrow l=0$, $\leftarrow j=1/2$

For a given j, there are $2j+1$ degenerate sublevels

$$\begin{cases} 2s_{1/2} \Rightarrow |2s_{1/2}, +\tfrac{1}{2}\rangle, \ |2s_{1/2}, -\tfrac{1}{2}\rangle \\ 2p_{1/2} \Rightarrow |2p_{1/2}, +\tfrac{1}{2}\rangle, \ |2p_{1/2}, -\tfrac{1}{2}\rangle \\ 2p_{3/2} \Rightarrow |2p_{3/2}, \tfrac{3}{2}\rangle, \ |2p_{3/2}, \tfrac{1}{2}\rangle, \ |2p_{3/2}, -\tfrac{1}{2}\rangle, \ |2p_{1/2}, -\tfrac{3}{2}\rangle \end{cases}$$

Since $\hat{H}_{int}$ acts only on the spatial degree of freedom, it will be useful to reëxpress the eigenstates above in terms of the "uncoupled" angular momentum basis. We did this in P.S. # 8, problem 2 (521 Fall 2000). The results were

$$|2s_{1/2}, \pm\tfrac{1}{2}\rangle = |2s, \overset{m_l}{0}\rangle \otimes |\overset{m_s}{\pm\tfrac{1}{2}}\rangle$$

$$|2p_{1/2}, \pm\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|2p, 0\rangle \otimes |\pm\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|2p, \pm 1\rangle \otimes |\mp\tfrac{1}{2}\rangle$$

$$|2p_{3/2}, \pm\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|2p, 0\rangle \otimes |\pm\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|2p, \pm 1\rangle \otimes |\mp\tfrac{1}{2}\rangle$$

$$|2p_{3/2}, \pm\tfrac{3}{2}\rangle = |2p, \pm 1\rangle \otimes |\pm\tfrac{1}{2}\rangle$$

(a) For weak fields $ea_0 E_z \gtrsim \Delta E_{Lamb}$, we can restrict our attention to the $(2s_{1/2}, 2p_{1/2})$ manifold

The matrix representation of $\hat{H}_{int}$ is block diagonal with no off diagonal ~~elem~~ elements between different m_j as we will see below

Consider $m_j = 1/2$, 2 dim space

$$\hat{H}_0 + \hat{H}_{int} \doteq \begin{array}{c} \begin{bmatrix} \Delta E_L & \epsilon \\ \epsilon^* & 0 \end{bmatrix} \\ |2s_{1/2}\rangle \;\; |2p_{1/2}\rangle \end{array}$$

where ΔE_L = Lamb shift

$$\epsilon = \langle 2p_{1/2}, \tfrac{1}{2} | \hat{H}_{int} | 2s_{1/2}, 1/2 \rangle$$

To calculate ϵ, we use the uncoupled representation above:

$$\epsilon = \underset{-eE_z}{\wedge} \sqrt{\tfrac{1}{3}} \langle 2p, 0 | \hat{z} | 2s, 0 \rangle \overset{1}{\cancel{\langle +\tfrac{1}{2} | +\tfrac{1}{2} \rangle}} - \sqrt{\tfrac{2}{3}} \langle 2p, 1 | \hat{z} | 2s, 0 \rangle \overset{0}{\cancel{\langle -\tfrac{1}{2} | +\tfrac{1}{2} \rangle}}$$

orthogonal spin

From class $\langle 2p, 0 | \hat{z} | 2s, 0 \rangle = -3a_0$

$$\Rightarrow \boxed{\epsilon = \mp\sqrt{3}\, e a_0 E_z} \quad \text{(real)}$$

Diagonalize $\hat{H} = \begin{bmatrix} \Delta E_L & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$= \frac{\Delta E_L}{2} \hat{1} + \frac{\Delta E_L}{2} \hat{\sigma}_z + \epsilon\, \hat{\sigma}_x$$

Eigenvalues $\boxed{E_\pm = \frac{\Delta E_L}{2} \pm \sqrt{\frac{(\Delta E_L)^2}{4} + \epsilon^2}}$

$$\boxed{\begin{array}{l}\text{Eigenvectors: } |\pm\rangle = \cos\left(\frac{\Theta}{2}\right)|2p_{1/2}\rangle \pm \sin\left(\frac{\Theta}{2}\right)|2s_{1/2}\rangle \\ \quad \text{where } \tan\Theta = \frac{2\epsilon}{\Delta E_L} \quad \text{("mixing angle")} \\ \text{Note: ratio of coupling matrix element to energy separation}\end{array}}$$

New splitting between perturbed $2s_{1/2}$ and $2p_{1/2}$

$$\Delta E_L' = E_+ - E_- = \sqrt{(\Delta E_L)^2 + 4\epsilon^2}$$

Find electric field such that $\Delta E_L' = 2\Delta E_L$

$$\Rightarrow 4\epsilon^2 = 3(\Delta E_L)^2 \Rightarrow \epsilon = \frac{\sqrt{3}}{2}\Delta E_L$$

$$\Rightarrow \sqrt{3}\, e\, a_0 E_z = \frac{\sqrt{3}}{2}\Delta E_L$$

$$\Rightarrow \boxed{E_z = \frac{1}{2ea_0}\Delta E_L}$$

Now for the numbers. Remember, we are using c.g.s. units. The easiest thing to do is express ΔE_L in electron volts, so that $\frac{\Delta E_L}{e}$ is in volts.

Conversion: Planck's constant $h = 4.14 \times 10^{-15}$ eV·s

$$\Rightarrow \Delta E_L = (10^9\ \text{Hz})(4.14 \times 10^{-15}\ \text{eV s}) = 4.14 \times 10^{-6}\ \text{eV}$$

$$a_0 = 0.5 \times 10^{-8}\ \text{cm} \quad (0.5\ \text{Å})$$

$$\therefore \boxed{E_z = \frac{4.14 \times 10^{-6}\ \text{V}}{10^{-8}\ \text{cm}} = 414\ \text{V/cm}}$$

What about the other m_j sub states?

- No off-diagonal matrix elements between different m_j

Proof $\langle 2s_{1/2}\tfrac{1}{2}|\hat{H}_{int}|2p_{1/2},-\tfrac{1}{2}\rangle$

$= +eE_z\left[\left(\langle 2s,0|\langle\tfrac{1}{2}|\right)\hat{z}\left(|2p,0\rangle|-\tfrac{1}{2}\rangle\right) - \sqrt{\tfrac{2}{3}}\left(\langle 2s,0|\langle\tfrac{1}{2}|\right)\hat{z}\left(|2p,-1\rangle|\tfrac{1}{2}\rangle\right)\right]$

$= +eE_z\left[\frac{1}{\sqrt{3}}\langle 2s,0|\hat{z}|2p,0\rangle \underbrace{\langle\tfrac{1}{2}|-\tfrac{1}{2}\rangle}_{0} - \sqrt{\tfrac{2}{3}}\underbrace{\langle 2s,0|\hat{z}|2p,-1\rangle}_{0}\langle\tfrac{1}{2}|\tfrac{1}{2}\rangle\right]$

$= 0$ ✓ and similarly for $\langle 2s\,-\tfrac{1}{2}|\hat{H}_{int}|2p_{1/2},\tfrac{1}{2}\rangle$

- The 2×2 matrix representation for $m_j = -1/2$ is the <u>same</u> as for $m_j = 1/2$ (try this yourself).

Thus in the 4-dim subspace of $(2s_{1/2}, 2p_{1/2})$ the representation of $\hat{H}$ is block-diagonal, with two degenerate sub-blocks

$$\hat{H} = \left[\begin{array}{cc|cc} E_L & \epsilon & & \\ \epsilon & 0 & & \\ \hline & & E_L & \epsilon \\ & & \epsilon & 0 \end{array}\right] \begin{array}{l} m_j = 1/2 \\ \\ m_j = -1/2 \end{array}$$

$\boxed{\text{Thus, the eigenvalue we found above are \underline{doubly} degenerate}}$

(b) Consider $ea_0 E_z \gtrsim \Delta E_{FS} \Rightarrow$ include all states in $n=2$

Again $\hat{H}$ is block diagonal, with no off-diagonal matrix element between different m_j. These ~~ones~~ block are also doubly degenerate for $\pm m_j$. As in class, there are no $p \to p$ matrix elements.

We must thus diagonalize the following 3x3 matrix:

$$\begin{array}{cc} \hat{H} \doteq \begin{bmatrix} \Delta E_L & \epsilon & \beta \\ \epsilon & 0 & 0 \\ \beta & 0 & \Delta E_{FS} \end{bmatrix} & m_j = \pm 1/2 \\ \quad |2s_{1/2}\rangle \quad |2p_{1/2}\rangle \quad |2p_{3/2}\rangle & \end{array}$$

note the $|2p_{3/2}, m_j = \pm 3/2\rangle$ is unperturbed

Here $\beta = \langle 2p_{3/2}, 1/2 | \hat{H}_{int} | 2s_{1/2}, 1/2\rangle$ (real)

$$= -eE_z \left[\sqrt{\tfrac{2}{3}} \underbrace{\langle 2p, 0 | \hat{z} | 2s\, 0\rangle}_{-3a_0} \underbrace{\langle \tfrac{1}{2} | \tfrac{1}{2}\rangle}_{=1} + \sqrt{\tfrac{1}{3}} \langle 2p, 1 | \hat{z} | 2s, 0\rangle \cancel{\langle -1/2 | 1/2\rangle} \right.$$

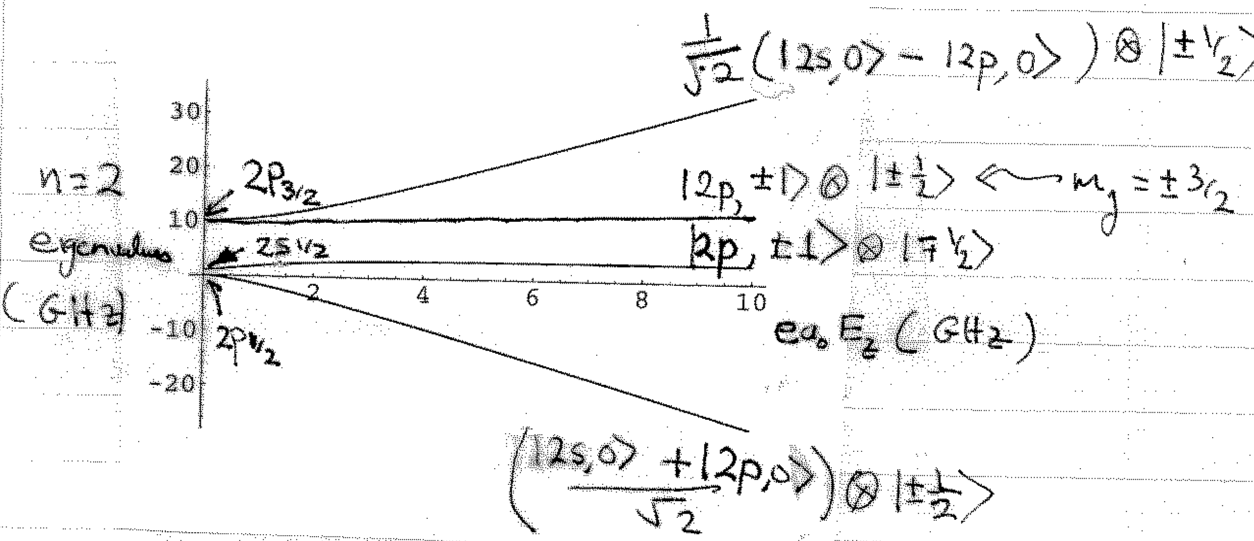
0

$$\Rightarrow \boxed{\beta = \sqrt{6}\, ea_0 E_z}$$

$$\Rightarrow \hat{H} \doteq \Delta E_L \begin{bmatrix} 1 & \sqrt{3}x & \sqrt{6}x \\ \sqrt{3}x & 0 & 0 \\ \sqrt{6}x & 0 & 10 \end{bmatrix} \qquad x \equiv \frac{ea_0 E_z}{\Delta E_L}$$

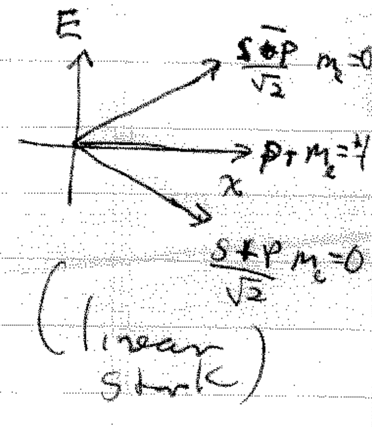
$\Delta E_L = 1\,\mathrm{GHz}$

Solving for the eigenvalues numerically in the range $0 < x < 10$



(c) Asymptotic behavior. Note for small x we recover the behavior of part (a) (the level $|2p_{3/2}\rangle$ is too far away). For sufficiently large x the fine-structure is negligible and we recover the simple linear stark shift discussed in class. That we cover the expected eigenvectors can be seen in the large x limit setting $\frac{\Delta E_{FS}}{x} = \frac{\Delta E_L}{x} = 0$

$$x \gg 1 \Rightarrow \hat{H} \doteq -x \begin{bmatrix} 0 & \sqrt{3} & \sqrt{6} \\ \sqrt{6} & 0 & 0 \\ \sqrt{3} & 0 & 0 \end{bmatrix}$$



$\Rightarrow$ Eigenvalues $\{-3x, 0, 3x\}$

Eigenvectors (next page)

$$\text{Eigenvectors:} \quad \left(\begin{array}{c} |e_1\rangle \\ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \end{array}, \begin{array}{c} |e_2\rangle \\ \begin{bmatrix} 0 \\ -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \end{array}, \begin{array}{c} |e_3\rangle \\ \begin{bmatrix} +\sqrt{\frac{1}{2}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \end{array}\right)$$

(up to arbitrary overall phase)

in the ordered basis $(|2s_{1/2}\rangle, |2p_{1/2}\rangle, |2p_{3/2}\rangle)$ $m_j = 1/2$

$$\Rightarrow |e_1\rangle = -\frac{1}{\sqrt{2}}|2s_{1/2}\rangle + \frac{1}{\sqrt{6}}|2p_{1/2}\rangle + \frac{1}{\sqrt{3}}|2p_{3/2}\rangle$$

$$= -\frac{1}{\sqrt{2}}|2s,0\rangle|1/2\rangle + \frac{1}{3\sqrt{2}}|2p,0\rangle|1/2\rangle - \cancel{\frac{1}{3}|2p,1\rangle|-1/2\rangle}$$
$$+ \frac{2}{3\sqrt{2}}|2p,0\rangle|1/2\rangle + \cancel{\frac{1}{3}|2p,1\rangle|-1/2\rangle}$$

$$\boxed{|e_1\rangle = -\frac{1}{\sqrt{2}}\left(|2s,0\rangle - |2p,0\rangle\right) \otimes |\tfrac{1}{2}\rangle} \checkmark$$

$$|e_2\rangle = -\sqrt{\frac{2}{3}}|2p_{1/2}\rangle + \frac{1}{\sqrt{3}}|2p_{3/2}\rangle = -\frac{\sqrt{2}}{3}|2p,0\rangle|1/2\rangle + \frac{2}{3}|2p,1\rangle|-1/2\rangle$$
$$+ \frac{\sqrt{2}}{3}|2p,0\rangle|1/2\rangle + \frac{1}{3}|2p,1\rangle|-1/2\rangle$$

$$\boxed{|e_2\rangle = |2p,1\rangle \otimes |-1/2\rangle} \checkmark$$

Same procedure $\Rightarrow$ $\boxed{|e_3\rangle = \frac{1}{\sqrt{2}}\left(|2s,0\rangle + |2p,0\rangle\right) \otimes |1/2\rangle}$ $\checkmark$

Note the $m_j = -1/2$ are the same asymptotes @ $m_j \to -m_j$

the $m_j = -3/2$ asymptote are flat throughout

and yeild the remaining states $|2p,\pm1\rangle \otimes |\pm 1/2\rangle$