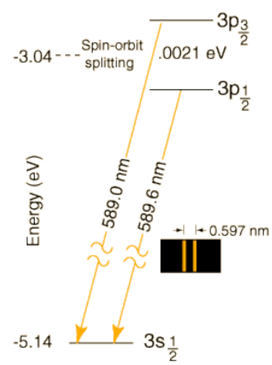
Physics 522. Quantum Mechanics II

Problem Set #9

Due **Thursday**, April 7, 2016

Problem 1: Hyperfine Structure of ^{23}Na (15 points).

Sodium, being an alkali, has one valence electron and thus has an energy level structure similar to hydrogen. The main difference is that the core now consists a nucleus screened by closed shell electrons. Adding electron spin leads to "fine structure" splitting. The energy level spectrum of the ground and first excited state then appears as below.



These two lines are known as the "sodium doublet". The transition $3s_{1/2} \to 3p_{1/2}$ is known as "D1" and $3s_{1/2} \to 3p_{3/2}$ known as "D2". All the alkalis have this basic structure.

(a) Consider the ^{23}Na isotope which has a nuclear spin quantum number I=3/2. Find *all* the possible hyperfine states for the ground electronic state and the excited states doublet. Express them as superpositions of the uncoupled basis vectors for electron and nucleus $|j,m_j\rangle \otimes |I,m_I\rangle$. You may use any source to find the Clebsch-Gordan coefficients (you need not calculate them). Is the new basis orthonomal?

(b) Check your answer for the $3p_{3/2}(F=3)$ state by directly apply the lowering operator to the "stretched state"

(c) Show that the dipole matrix element (dipole along z) for the three possible transitions $|3s_{1/2}(F=1,M_F=0)\rangle \to |3p_{1/2}(F=1,M'_F)\rangle$ vanish unless $M'_F=0$.

Problem 2: Zeeman effect in the ground state of hydrogen (20 points)

In the 1s state of hydrogen, the interaction of the atom with an external magnetic field is due solely to the spins since the orbital angular momentum is zero. Including the hyperfine interactions, the Hamiltonian is

$$\hat{H} = A\hat{\mathbf{i}} \cdot \hat{\mathbf{s}} + g_e \mu_B \mathbf{B} \cdot \hat{\mathbf{s}} - g_p \mu_N \mathbf{B} \cdot \hat{\mathbf{i}} ,$$

where $g_e = 2$ ($g_p \approx 5.6$) is the electron (proton) g-factor, μ_B (μ_N) is the Bohr (nuclear) magneton, and A is the hyperfine constant. Here the electron and proton spin operators are measured in units of $\hbar$.

(a) In weak field limit, $\mu_B B << A$, the interaction with the magnetic field can be treated as a perturbation to the hyperfine interaction. What strength of magnetic field is "weak", according to this condition in the ground state of hydrogen? What are the "good quantum numbers" in this regime? Find the eigenstates and shifts in the energy levels per Gauss of magnetic field, to lowest nonvanishing order in perturbation theory for weak fields.

(b) In the strong field limit (also known as Pashen-Back regime), $\mu_B B >> A$, the hyperfine interaction can be treated as a perturbation to the Zeeman Hamiltonian. What are the "good quantum numbers" in this regime? Find the eigenstates and shifts in the energy levels per Gauss of magnetic field to lowest nonvanishing order in perturbation theory for strong fields.

(c) This Hamiltonian can be solved *exactly* in the subspace of the 1s state. Show that $\hat{f}_z = \hat{i}_z + \hat{s}_z$ commutes with the total Hamiltonian. Use this to block-diagonalize the Hamiltonian. Then diagonalize each block in turn. Let m_f be the eigenvalue of $\hat{f}_z$. Show that the exact eigenvalues are of the form,

$$E_{\pm}(m_f) = -g_p \mu_N B m_f - \frac{A}{4} \pm \frac{A}{2}\sqrt{1 + 2m_f x + x^2}$$

where $x = (g_p \mu_B + g_e \mu_B)B / A$. This is known as the "Breit-Rabi formula".

This formula is interpreted as

$$E_{\pm}(m_f = 0) = -\frac{A}{4} \pm \frac{A}{2}\sqrt{1+x^2},$$

$$E(m_f = 1) = -g_p \mu_N B - \frac{A}{4} + \frac{A}{2}\sqrt{1+2x+x^2}, \; E(m_f = -1) = +g_p \mu_N B - \frac{A}{4} - \frac{A}{2}\sqrt{1-2x+x^2}$$

(d) Plot the four energy levels as function of x (neglect the small contribution of the nuclear magneton). Label them in the strong and weak limits. Comment on the behavior of these curves.

(e) Show that in the weak and strong limits, the analytic expression agrees with your answers in parts (a) and (b).