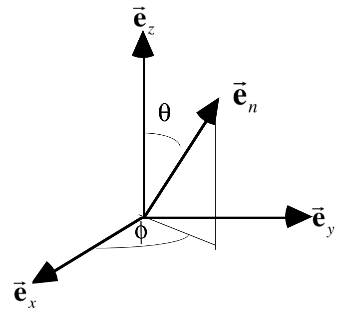
Physics 522, Spring 2016

Problem Set #10

Due: Thursday Apr. 14, 2016 @ 5PM

Problem 1: Spin-1/2 , SU(2) and rotations (25 points)

(a) Consider a unit vector $\vec{\mathbf{e}}_n$, defined by angles θ and ϕ with respect to the polar axis *z*.



Argue that the ket, spin up along $\vec{\mathbf{e}}_n$, can be obtained through a sequence of SU(2) rotations, $\left|\uparrow_n\right\rangle = \hat{D}_z(\phi)\hat{D}_y(\theta)\left|\uparrow_z\right\rangle$, where $\hat{D}_{\mathbf{n}}(\alpha) = \exp\left(-i\frac{\alpha}{2}\mathbf{n}\cdot\hat{\vec{\sigma}}\right)$, and that up to an overall phase,

$\left|\uparrow_n\right\rangle = \cos\frac{\theta}{2}\left|\uparrow_z\right\rangle + e^{i\phi}\sin\frac{\theta}{2}\left|\downarrow_z\right\rangle$.

(b) Use the multiplication of the Pauli matrices to show

$$\hat{D}_z^\dagger(\phi)\hat{\sigma}_x\hat{D}_z(\phi) = \cos\phi\,\hat{\sigma}_x + \sin\phi\,\hat{\sigma}_y$$
$$\hat{D}_y^\dagger(\theta)\hat{\sigma}_z\hat{D}_y(\theta) = \cos\theta\,\hat{\sigma}_z - \sin\theta\,\hat{\sigma}_x$$

Is this what you expect?

(c) Consider an electron in a magnetic field with an intrinsic magnetic moment, $\hat{\vec{\mu}} = -2\mu_B\hat{\mathbf{S}}/\hbar$, where $\hat{\mathbf{S}}$ is the spin-1/2 angular momentum operator. When placed in a magnetic field **B**, the interaction energy is described by the Hamiltonian, $\hat{H} = -\hat{\vec{\mu}}\cdot\mathbf{B}$. Given $\left|\psi(0)\right\rangle = \left|\uparrow_z\right\rangle$, find $\left|\psi(t)\right\rangle$ for an arbitrary direction of **B.**

(d) Show that the Heisenberg equation of motion for the spin operator is

$$\frac{d\hat{\mathbf{S}}}{dt} = \vec{\Omega}\times\hat{\mathbf{S}}, \text{ where } \hbar\vec{\Omega} = 2\mu_B\mathbf{B}$$

Describe the physical motion of $\left\langle\hat{\mathbf{S}}(t)\right\rangle$.

(e) Now consider an atom with total angular momentum **F**. We will that when restricted to a subspace with total F, the magnetic dipole moment $\hat{\vec{\mu}} = -g_F \mu_B \hat{\mathbf{F}}/\hbar$, where g_F is known as the Landé g-factor. Describe the physical motion of $\langle \hat{\mathbf{F}}(t) \rangle$ is a magnetic field.

Problem 2: Landé Projection Theorem (20 Points)

The Landé Projection Theorem (LPT) is a special case of the Wigner-Eckart Theorem for the case that the initial and final state are the same angular momentum. It states:

$$\langle \alpha'; jm' | \hat{\mathbf{V}} | \alpha; jm \rangle = \frac{\langle \alpha'; j | \hat{\mathbf{J}} \cdot \hat{\mathbf{V}} | \alpha; j \rangle}{j(j+1)} \langle jm' | \hat{\mathbf{J}} | jm \rangle, \text{ where } \hat{\mathbf{V}} \text{ is a vector operator w.r.t. } \hat{\mathbf{J}}.$$

(a) Give a geometric interpretation of this in terms of a vector picture.
(b) To prove this theorem, take the following steps (do not give verbatim, Sakurai's derivation):

- (i) Show that $\langle \alpha'; j, m | \hat{\mathbf{J}} \cdot \hat{\mathbf{V}} | \alpha; j, m \rangle = \langle \alpha'; j \| J \| \alpha'; j \rangle \langle \alpha'; j \| V \| \alpha; j \rangle$, independent of m.
- (ii) Use this to show, $|\langle \alpha; j \| J \| \alpha; j \rangle|^2 = j(j+1)$ *independent of* α.
- (iii) Show that $\langle jm' | 1 q jm \rangle = \langle jm' | \hat{J}_q | jm \rangle / \sqrt{j(j+1)}$.
- (iv) Put it all together to prove the LPT.

(c) As an application of the LPT, consider the Zeeman interaction for hydrogen.

$$\hat{H}_{\text{int}} = -\hat{\vec{\mu}} \cdot \mathbf{B},$$

where the magnetic dipole operators is $\hat{\vec{\mu}} = -\mu_B (g_l \hat{\mathbf{L}} + g_s \hat{\mathbf{S}})$, with $g_l = 1, g_s = 2$.

If the Zeeman effect is small compared to the fine-structure, but ignoring hyperfine structure, use the LPT to show that in a state nL_J, the magnetic moment has the form,

$$\hat{\vec{\mu}} = -g_J \mu_B \hat{\vec{\mathbf{J}}}, \text{ where } g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)} \text{ is known as the Landé g-factor.}$$

Hint: Use $\mathbf{J} \cdot \mathbf{L} = \mathbf{L}^2 + \frac{1}{2}(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2)$ and $\mathbf{J} \cdot \mathbf{S} = \mathbf{S}^2 + \frac{1}{2}(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2)$.

(d) Estimate the magnetic field strength at which the Zeeman interaction becomes on the order of the fine structure splitting between the $2p_{1/2}$ and $2p_{3/2}$ state in hydrogen.