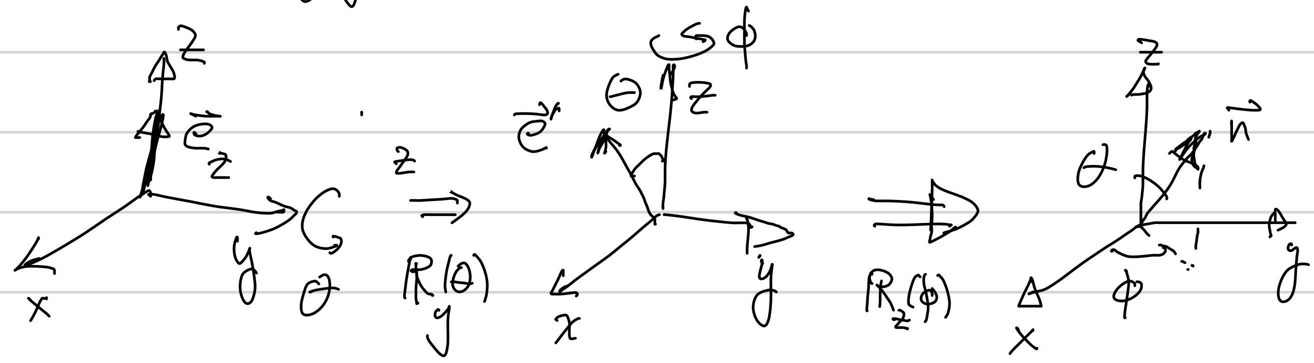
Physics 522: Quantum II
Problem Set #10: Solutions

Problem 1: Spin-1/2, SU(2) rotations

(a) Consider the direction on the sphere $\vec{n}$ defined by polar angles (θ,ϕ). We can obtain that unit vector by starting with a unit vector along the z-axis and then applying a series of rotations



Thus, up to some overall phase convention

$$|\uparrow_{\vec{n}}\rangle = \hat{D}_z(\phi)\,\hat{D}_y(\theta)\,|\uparrow_z\rangle$$

Now $\hat{D}_z(\phi) = \cos\frac{\phi}{2}\hat{\mathbb{1}} - i\sin\frac{\phi}{2}\hat{\sigma}_z$, $\quad \hat{D}_y(\theta) = \cos\frac{\theta}{2}\hat{\mathbb{1}} - i\sin\frac{\theta}{2}\hat{\sigma}_y$

$$\Rightarrow \hat{D}_z(\theta)\,\hat{D}_y(\theta) = \left(\cos\frac{\phi}{2}\hat{\mathbb{1}} - i\sin\frac{\phi}{2}\hat{\sigma}_z\right)\left(\cos\frac{\theta}{2}|\uparrow_z\rangle + \sin\frac{\theta}{2}|\downarrow_z\rangle\right)$$

$$= e^{-i\phi/2}\cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}|\downarrow_z\rangle = \underbrace{e^{-i\phi/2}}_{\text{neglect}}\left(\cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow_z\rangle\right)$$

Here we used $-i\hat{\sigma}_y = -i\left(\frac{\hat{\sigma}_+ - \hat{\sigma}_-}{i}\right) \Rightarrow -i\hat{\sigma}_y|\uparrow_z\rangle = |\downarrow_z\rangle$

$$\hat{\sigma}_z|\uparrow_z\rangle = |\uparrow_z\rangle, \quad \hat{\sigma}_z|\downarrow_z\rangle = -|\downarrow_z\rangle$$

(b) We know that $\hat{\vec{\sigma}}$ rotates as a vector under $\hat{D}(R_{\vec{n}}(\phi)) = e^{-i\frac{\phi}{2}\vec{n}\cdot\hat{\vec{\sigma}}}$

$$\Rightarrow \hat{D}^\dagger(R_{\vec{n}}(\phi))\,\hat{\sigma}_i\,\hat{D}(R_{\vec{n}}(\phi)) = R_{ij}(\vec{n},\phi)\,\hat{\sigma}_j$$

Rotation around z axis by ϕ, we expect $\hat{D}_z^\dagger(\phi)\,\hat{\sigma}_x\,\hat{D}_z(\phi) = \hat{\sigma}_x' = \cos\phi\,\hat{\sigma}_x + \sin\phi\,\hat{\sigma}_y$

Rotation around y axis by θ, we expect $\hat{D}_y^\dagger(\theta)\,\hat{\sigma}_z\,\hat{D}_y(\theta) = \hat{\sigma}_z' = \cos\theta\,\hat{\sigma}_z - \sin\theta\,\hat{\sigma}_x$

Check: recall $\hat{\sigma}_i \hat{\sigma}_j = i\epsilon_{ijk}\hat{\sigma}_k + \delta_{ij}\hat{\mathbb{1}}$

$$\hat{D}_z^{\dagger}(\phi)\,\hat{\sigma}_x\,\hat{D}_z(\phi) = \left(\cos\tfrac{\phi}{2}\hat{\mathbb{1}} + i\sin\tfrac{\phi}{2}\hat{\sigma}_z\right)\hat{\sigma}_x\left(\cos\tfrac{\phi}{2}\hat{\mathbb{1}} - i\sin\tfrac{\phi}{2}\hat{\sigma}_z\right)$$

$$= \left(\cos\tfrac{\phi}{2}\hat{\mathbb{1}} + i\sin\tfrac{\phi}{2}\hat{\sigma}_z\right)\left(\cos\tfrac{\phi}{2}\hat{\sigma}_x - \sin\tfrac{\phi}{2}\hat{\sigma}_y\right)$$

$$= \cos^2\tfrac{\phi}{2}\hat{\sigma}_x + \sin\tfrac{\phi}{2}\cos\tfrac{\phi}{2}\hat{\sigma}_y + \sin\tfrac{\phi}{2}\cos\tfrac{\phi}{2}\hat{\sigma}_y - \sin^2\tfrac{\phi}{2}\hat{\sigma}_x$$

$$= \left(\cos^2\tfrac{\phi}{2} - \sin^2\tfrac{\phi}{2}\right)\hat{\sigma}_x + \left(2\sin\tfrac{\phi}{2}\cos\tfrac{\phi}{2}\right)\hat{\sigma}_y$$

$$= \cos\phi\,\hat{\sigma}_x + \sin\phi\,\hat{\sigma}_y \quad ✓$$

$$\hat{D}_y^{\dagger}(\theta)\,\hat{\sigma}_z\,\hat{D}_y(\theta) = \left(\cos\tfrac{\theta}{2}\hat{\mathbb{1}} + i\sin\tfrac{\theta}{2}\hat{\sigma}_y\right)\hat{\sigma}_z\left(\cos\tfrac{\theta}{2}\hat{\mathbb{1}} - i\sin\tfrac{\theta}{2}\hat{\sigma}_y\right)$$

$$= \left(\cos\tfrac{\theta}{2}\hat{\mathbb{1}} + i\sin\tfrac{\theta}{2}\hat{\sigma}_y\right)\left(\cos\tfrac{\theta}{2}\hat{\sigma}_z - \sin\tfrac{\theta}{2}\hat{\sigma}_x\right)$$

$$= \left(\cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2}\right)\hat{\sigma}_z - 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\hat{\sigma}_x = \cos\theta\,\hat{\sigma}_z - \sin\theta\,\hat{\sigma}_x \quad ✓$$

(c) Given $|\psi(0)\rangle = |\uparrow_z\rangle$,

Hamiltonian $\hat{H} = -\hat{\vec{\mu}}\cdot\vec{B} = 2\mu_B \frac{\vec{S}}{\hbar}\cdot\vec{B} = 2\mu\vec{B}\cdot\frac{\hbar}{2}\hat{\vec{\sigma}} = \hbar\vec{\Omega}\cdot\frac{\hat{\vec{\sigma}}}{2}$, where $\hbar\vec{\Omega} = 2\mu_B\vec{B}$

Time evolved state $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\uparrow_z\rangle = e^{-i\frac{\Omega t}{2}\vec{n}\cdot\hat{\vec{\sigma}}}|\uparrow_z\rangle$, $\vec{\Omega} \equiv \Omega\vec{n}$

$\vec{n} = \vec{B}/|\vec{B}|$

$$\Rightarrow |\psi(t)\rangle = \left[\cos\left(\tfrac{\Omega t}{2}\right)\hat{\mathbb{1}} - i\sin\left(\tfrac{\Omega t}{2}\right)\vec{n}\cdot\hat{\vec{\sigma}}\right]|\uparrow_z\rangle$$

$$= \left[\cos\left(\tfrac{\Omega t}{2}\right)\hat{\mathbb{1}} - \sin\left(\tfrac{\Omega t}{2}\right)\left(n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z\right)\right]|\uparrow_z\rangle$$

$$\boxed{\begin{aligned}&\Rightarrow |\psi(t)\rangle = \left(\cos\tfrac{\Omega t}{2} + n_z\sin\tfrac{\Omega t}{2}\right)|\uparrow_z\rangle - (n_x + in_y)\sin\tfrac{\Omega t}{2}|\downarrow_z\rangle \\ &\Omega = \frac{2\mu_B|\vec{B}|}{\hbar} \qquad n_x = \frac{B_x}{|\vec{B}|},\ n_y = \frac{B_y}{|\vec{B}|},\ n_z = \frac{B_z}{|\vec{B}|}\end{aligned}}$$

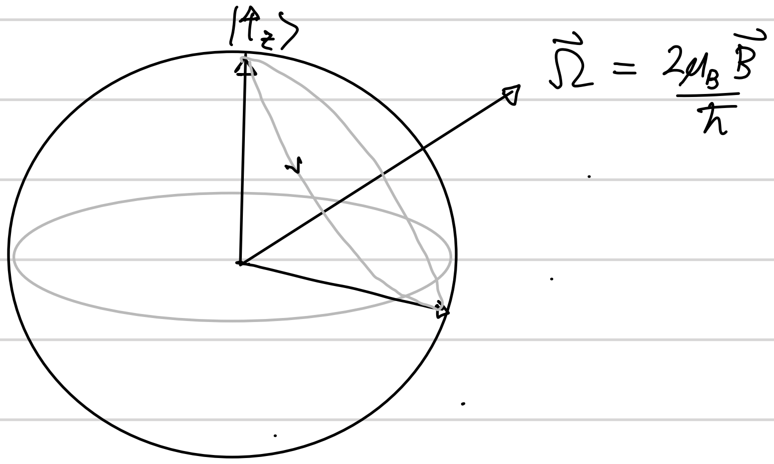
This is known as the "Rabi" solution

(d) We can easily solve for the motion of $\langle\vec{S}\rangle(t)$ using the Heisenberg picture.

$$\frac{d}{dt}\hat{S}_i(t) = \frac{-i}{\hbar}[\hat{S}_i, \hat{H}] = \frac{-i}{\hbar}[\hat{S}_i, \vec{\Omega}\cdot\vec{S}] = \frac{-i}{\hbar}\Omega_j[\hat{S}_i, \hat{S}_j] = \frac{-i}{\hbar}\Omega_j\, i\hbar\epsilon_{ijk}\hat{S}_k$$
$$= \epsilon_{ijk}\Omega_j\hat{S}_k = (\vec{\Omega}\times\hat{\vec{S}})_i$$

$$\Rightarrow \boxed{\frac{d}{dt}\langle\hat{\vec{S}}\rangle = \vec{\Omega}\times\langle\hat{\vec{S}}\rangle}$$

This is the equation of a gyroscope, representing the Larmor precession of $\langle\vec{S}\rangle$ about $\vec{\Omega}$



(e) Now we consider a magnetic moment defined by $\hat{\vec{\mu}} = -g_F\mu_B\vec{F}/\hbar$ where $\vec{F}$ is the total angular momentum.

Then, exactly as before $\frac{d\hat{F}_i}{dt} = \frac{-i}{\hbar}[\hat{F}_i, \hat{H}] = \frac{-i}{\hbar}[\hat{F}_i, \vec{\Omega}\cdot\hat{\vec{F}}]$, where $\vec{\Omega} = \frac{g_F\mu_B\vec{B}}{\hbar}$

$$\Rightarrow \frac{d\langle\hat{\vec{F}}\rangle}{dt} = \vec{\Omega}\times\langle\hat{\vec{F}}\rangle.$$ The total spin will Larmor precess.

It's evolution is described by an <u>SU(2) rotation</u>, even for spin $F > \frac{1}{2}$.

$\hat{U}(t) = e^{-i\vec{\Omega}t\cdot\frac{\hat{\vec{F}}}{\hbar}}$: Rotation operator $\Rightarrow$ on space $\{|F, M_F\rangle\}$ is an irrep of SU(2).

Problem 2. The Landé Projection theorem

The L.P.T.: $\langle \alpha'; j m' | \hat{\vec{V}} | \alpha; j m \rangle = \dfrac{\langle \alpha'; j | \hat{\vec{J}} \cdot \hat{\vec{V}} | \alpha; j \rangle}{j(j+1)} \langle \alpha'; j m' | \vec{J} | \alpha, j m \rangle$

(a) The L.P.T. states mathematically that all matrix elements of a vector operator $\hat{\vec{V}}$ restricted to states with eigenvalue j are proportional to matrix element of $\hat{\vec{J}}$

i.e., in this subspace $\quad \hat{\vec{V}} = C(\alpha' \alpha; j) \, \hat{\vec{J}}$

constant depending only on j and other non-geometric q-numbers

Geometrically, if we imagine $\vec{J}$ to be a vector, classically the vector $\vec{V}$ along $\vec{J}$ is

$$(\vec{V} \cdot \vec{e}_J) \vec{e}_J \quad \text{where} \quad \vec{e}_J = \frac{\vec{J}}{|\vec{J}|}$$

$$= \left(\frac{\vec{V} \cdot \vec{J}}{|\vec{J}|^2} \right) \vec{J}$$

Here with $|\vec{J}|^2 = j(j+1)$, the "classical" projection is

$$\vec{V}\Big|_{\vec{J}} = \underbrace{\left[\frac{\vec{V} \cdot \vec{J}}{j(j+1)} \right]}_{\text{"Constant"}} \vec{J}$$

(b) Proof of the L.P.T.

(i) Consider $\langle \alpha'; jm | \hat{\vec{J}} \cdot \hat{\vec{V}} | \alpha j m \rangle = \sum_q (-1)^q \langle \alpha'; jm | \hat{J}_{q} \hat{V}_{-q} | \alpha; jm \rangle$

Insert complete set $\sum_{m'} |\alpha' j m'\rangle\langle \alpha' j m|$

$$= \sum_{q,m'} (-1)^q \langle \alpha'; jm | \hat{J}_q | \alpha'; jm' \rangle \langle \alpha'; jm' | \hat{V}_{-q} | \alpha; jm \rangle$$

By W.E.T.

$$= \sum_{q m'} (-1)^q \langle jm | 1 q\, jm' \rangle \langle jm' | 1 -q\, jm \rangle \langle \alpha' j \| J \| \alpha' j \rangle \langle \alpha' j \| V \| \alpha j \rangle$$

Aside $\langle jm' | 1 -q\, jm \rangle = (-1)^q \langle 1 q\, jm' | jm \rangle$ (Exchange rules)

$$= \sum_{qm'} \underbrace{\langle jm | 1 q\, jm' \rangle \langle 1 q\, jm' | jm \rangle}_{=1} \langle \alpha' j \| J \| \alpha' j \rangle \langle \alpha' j \| V \| \alpha j \rangle$$

$$= \langle \alpha' j \| J \| \alpha' j \rangle \langle \alpha' j \| V \| \alpha j \rangle$$ independent of m (as it should be for a scalar)

(ii) For $\hat{\vec{V}} = \hat{\vec{J}}$, $\alpha = \alpha'$ $\quad \langle \alpha; j | \vec{J}^2 | \alpha j \rangle = |\langle \alpha j \| J \| \alpha j \rangle|^2$

$$\langle \alpha; j | \vec{J}^2 | \alpha j \rangle = j(j+1) \checkmark$$

Independent of α

(iii) Apply W.E.T. to $\hat{J}_q$

$$\langle jm' | \hat{J}_q | jm \rangle = \langle j \| J \| j \rangle \langle jm' | 1 q\, jm \rangle$$

$$\Rightarrow \quad \langle jm' | 1 q\, jm \rangle = \frac{\langle jm' | \hat{J}_q | jm \rangle}{\langle j \| J \| j \rangle} = \frac{\langle jm' | \hat{J}_q | jm \rangle}{\sqrt{j(j+1)}}$$

(iv) Putting it all together

From W.E.T. $\langle \alpha' j m' | \hat{V}_q | \alpha j m \rangle = \langle \alpha' j \| V \| \alpha j \rangle \langle j m' | 1 q j m \rangle$

$$\langle \alpha' j \| V \| \alpha j \rangle = \frac{\langle \alpha' j | \hat{\vec{J}} \cdot \hat{\vec{V}} | \alpha j \rangle}{\langle j \| J \| j \rangle} \qquad \langle j m' | 1 q j m \rangle = \frac{\langle \alpha' j m' | \hat{J}_q | \alpha j m \rangle}{\langle j \| J \| j \rangle}$$

True for any component q

$$\Rightarrow \boxed{\langle \alpha' j m' | \hat{\vec{V}} | \alpha j m \rangle = \frac{\langle \alpha' j | \hat{\vec{J}} \cdot \hat{\vec{V}} | \alpha j \rangle}{j(j+1)} \langle \alpha' j m' | \hat{\vec{J}} | \alpha j m \rangle}$$

(c) Application: the Landé g-factor

The magnetic dipole moment for the electron is

$$\hat{\vec{\mu}} = -\mu_B \left(g_\ell \hat{\vec{L}} + g_s \hat{\vec{S}} \right) \qquad g_\ell = 1 \qquad g_s = 2$$

When restricted to a manifold with definite $\vec{J} = \vec{L} + \vec{S}$

(i.e. the Zeeman interaction is small compared to fine structure)

$$\hat{\vec{\mu}} = \frac{\langle \alpha J | \hat{\vec{\mu}} \cdot \hat{\vec{J}} | \alpha J \rangle}{J(J+1)} \hat{\vec{J}} = -\mu_B \, g_J \, \hat{\vec{J}}$$

where $g_J = -\dfrac{\langle \alpha J | \hat{\vec{\mu}} \cdot \hat{\vec{J}} | \alpha J \rangle}{\mu_B J(J+1)} = \dfrac{\langle \alpha J | \left(\hat{\vec{L}} \cdot \hat{\vec{J}} + 2 \hat{\vec{S}} \cdot \hat{\vec{J}} \right) | \alpha J \rangle}{J(J+1)}$

Aside: $\hat{\vec{L}} \cdot \hat{\vec{J}} = \hat{\vec{L}} \cdot (\hat{\vec{L}} + \hat{\vec{S}}) = \hat{\vec{L}}^2 + \hat{\vec{L}} \cdot \hat{\vec{S}}$

$\hat{\vec{S}} \cdot \hat{\vec{J}} = \hat{\vec{S}} \cdot (\hat{\vec{L}} + \hat{\vec{S}}) = \hat{\vec{S}}^2 + \hat{\vec{L}} \cdot \hat{\vec{S}}$

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Aside continued: $\hat{J}^2 = |\hat{L}+\hat{S}|^2 = \hat{L} + \hat{S}^2 + 2\hat{L}\cdot\hat{S}$

$$\Rightarrow \hat{L}\cdot\hat{S} = \frac{1}{2}(\hat{J}^2 - \hat{L}^2 - \hat{S}^2)$$

$$\Rightarrow \hat{L}\cdot\hat{J} = \hat{L}^2 + \frac{1}{2}(\hat{J}^2 - \hat{L}^2 - \hat{S}^2)$$

$$\hat{S}\cdot\hat{J} = \hat{S}^2 + \frac{1}{2}(\hat{J}^2 - \hat{L}^2 - \hat{S}^2)$$

$$\Rightarrow g_J = \frac{L(L+1) + 2S(S+1) + \frac{3}{2}(J(J+1) - L(L+1) - S(S+1))}{J(J+1)}$$

where I have replaced the operators $\hat{J}^2, \hat{L}^2, \hat{S}^2$ by their eigenvalues associated with the good quantum numbers of that level.

For the electron $S = 1/2 \Rightarrow S(S+1) = \frac{3}{4}$

Simplify $\Rightarrow$ $$\boxed{g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)} = \frac{3}{2} + \frac{3}{8J(J+1)} - \frac{L(L+1)}{2J(J+1)}}$$

Note when $L=0$ $g_J = 2$ as expected

(d) For the $2P_{1/2}$ state $L=1$ $J=1/2$ $\Rightarrow g_J(2P_{1/2}) = 2/3$

$2P_{3/2}$ state $L=1$ $J=3/2$ $\Rightarrow g_J(2P_{3/2}) = 4/3$

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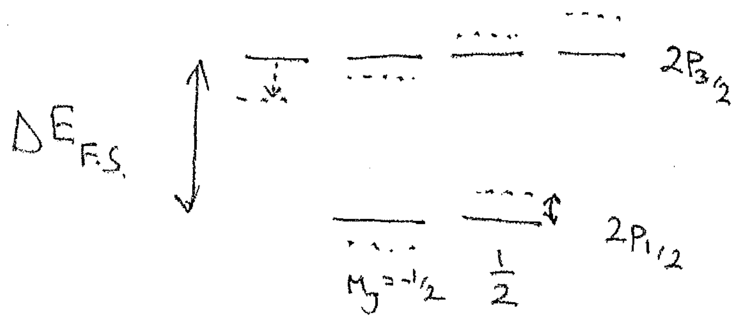
The Zeeman interaction acts only in the manifold nL_J when the shifts are small compared to the fine-structure splitting.



Zeeman shift:

$$m_J\, g_J\, (\mu_B\, B) = \Delta E_Z$$

Require

$$\frac{3}{2}\, g_J(2P_{3/2})\, \mu_B B + \frac{1}{2}\, g_J(2P_{1/2})\, \mu_B B \ll \Delta E_{F.S.}$$

$$\Longrightarrow \mu_B B \ll \frac{3}{7}\, \Delta E_{F.S.}$$

The fine-structure splitting between $2P_{3/2}$ and $2P_{1/2}$ is $\Delta E_{F.S} = (11\text{ GHz})\, h$, and $\mu_B = (1.4\text{ MHz/Gauss})\, h$

$\Longrightarrow$ Require $\boxed{B \ll \left(\frac{3}{7}\right)\left(\frac{11\times10^{3}}{1.4}\right) G = 3.4\text{ MGauss} = 3.4\text{ Tesla}}$

A huge field!