

Physics 522, Spring 2016

Problem Set #11

Due: Thursday Apr. 21, 2016 @ 5PM

Problem 1: Natural lifetimes of Hydrogen (20 Points)

Though in the absence of any perturbation, an atom in the excited state will stay there forever (it is a stationary state), in reality it will spontaneously decay to the ground state. Fundamentally this occurs because the atom is always perturbed by ‘vacuum fluctuations’ in the electromagnetic field, together with radiation reaction. We found that the spontaneous emission rate on a dipole allowed transition from initial excited state $|\psi_e\rangle$ to all allowed ground states $|\psi_g\rangle$ is given by the Einstein-A coefficient,

$$\Gamma = A = \frac{4}{3\hbar} k^3 \sum_g \left| \langle \psi_g | \hat{\mathbf{d}} | \psi_e \rangle \right|^2, \text{ where } k = \omega_{eg} / c \text{ is the emitted photon's wave number.}$$

Consider now hydrogen including fine-structure. For a given sublevel, the spontaneous emission rate is

$$\Gamma_{(nLJM_J) \rightarrow (n'L'J'M'_J)} = \frac{4}{3\hbar} k^3 \sum_{M'_J} \left| \langle n'L'J'M'_J | \hat{\mathbf{d}} | nLJM_J \rangle \right|^2.$$

(a) Show that the spontaneous emission rate is *independent* of the initial M_J . Explain this result physically.

(b) Calculate the lifetime ($\tau=1/\Gamma$) of the $2p_{1/2}$ state in seconds.

Problem 2: Light-shift for multilevel atoms (20 points)

We found the AC-Stark (light shift) for the case of a two-level atom driven by a monochromatic field. In this problem we want to look at this phenomenon in a more general context, including arbitrary polarization of the electric field, and atoms with multiple sublevels.

Consider then a general monochromatic electric field $\mathbf{E}(\mathbf{x}, t) = \text{Re}(\mathbf{E}(\mathbf{x})e^{-i\omega_L t})$, driving an atom near resonance on the transition, $|g; J_g\rangle \rightarrow |e; J_e\rangle$, where the ground and excited manifolds are each described by some total angular momentum J with degeneracy $2J+1$. The generalization of the AC-Stark shift is now the light-shift operator acting on the $2J_g + 1$ dimensional ground manifold:

$$\hat{V}_{LS}(\mathbf{x}) = -\frac{1}{4} \mathbf{E}^*(\mathbf{x}) \cdot \hat{\alpha} \cdot \mathbf{E}(\mathbf{x}).$$

Here $\hat{\alpha} = -\frac{\hat{\mathbf{d}}_{ge} \hat{\mathbf{d}}_{eg}}{\hbar\Delta}$ is the atomic polarizability tensor operator, where $\hat{\mathbf{d}}_{eg} \equiv \hat{P}_e \hat{\mathbf{d}} \hat{P}_g$ is the dipole operator, projected between the ground and excited manifolds; the projector onto the excited manifold is, $\hat{P}_e = \sum_{M_e=-J_e}^{J_e} |e; J_e, M_e\rangle\langle e; J_e, M_e|$, and similarly for the ground.

(a) By expanding the dipole operator in the spherical basis, show that the polarizability operator can be written,

$$\hat{\alpha} = \tilde{\alpha} \left(\sum_{q, M_g} |C_{M_g}^{M_g+q}|^2 \bar{\mathbf{e}}_q |g; J_g, M_g\rangle\langle g; J_g, M_g| \bar{\mathbf{e}}_q^* + \sum_{q \neq q', M_g} C_{M_g+q-q'}^{M_g+q} C_{M_g}^{M_g+q} \bar{\mathbf{e}}_{q'} |g; J_g, M_g+q-q'\rangle\langle g; J_g, M_g| \bar{\mathbf{e}}_q^* \right),$$

$$\text{where } \tilde{\alpha} \equiv -\frac{|\langle e; J_e || d || g; J_g \rangle|^2}{\hbar\Delta} \text{ and } C_{M_g}^{M_e} \equiv \langle J_e M_e | 1q J_g M_g \rangle.$$

Explain physically, using dipole selection rules, the meaning of the expression for $\hat{\alpha}$.

(b) Consider a polarized plane wave, with complex amplitude of the form, $\mathbf{E}(\mathbf{x}) = E_1 \bar{\mathbf{e}}_L e^{i\mathbf{k}\cdot\mathbf{x}}$ where E_1 is the amplitude and $\bar{\mathbf{e}}_L$ the polarization (possibly complex). For an atom driven on the transition $|g; J_g = 1\rangle \rightarrow |e; J_e = 2\rangle$ and the cases (i) linear polarization along z , (ii) positive helicity polarization with \mathbf{k} along z , (iii) linear polarization along x , find the eigenvalues and eigenvectors of the light-shift operator. Express the eigenvalues in units of $V_1 = -\frac{1}{4} \tilde{\alpha} |E_1|^2$.

Please *comment* on what you find for cases (i) and (iii). Repeat for $|g; J_g = 1/2\rangle \rightarrow |e; J_e = 3/2\rangle$ and **comment**.

(c) **Extra Credit: 5 points**

A deeper insight into the light-shift potential can be seen by expressing the polarizability operator in terms of irreducible tensors. Verify that the total light shift is the sum of scalar, vector, and rank-2 irreducible tensor interaction,

$$\hat{V}_{LS} = -\frac{1}{4} \left(|\mathbf{E}(\mathbf{x})|^2 \hat{\alpha}^{(0)} + (\mathbf{E}^*(\mathbf{x}) \times \mathbf{E}(\mathbf{x})) \cdot \hat{\alpha}^{(1)} + \mathbf{E}^*(\mathbf{x}) \cdot \hat{\alpha}^{(2)} \cdot \mathbf{E}(\mathbf{x}) \right),$$

$$\text{where } \hat{\alpha}^{(0)} = \frac{\hat{\mathbf{d}}_{ge} \cdot \hat{\mathbf{d}}_{eg}}{-3\hbar\Delta}, \hat{\alpha}^{(1)} = \frac{\hat{\mathbf{d}}_{ge} \times \hat{\mathbf{d}}_{eg}}{-2\hbar\Delta}, \hat{\alpha}_{ij}^{(2)} = \frac{\hat{\mathbf{d}}_{ge}^i \hat{\mathbf{d}}_{eg}^j + \hat{\mathbf{d}}_{ge}^j \hat{\mathbf{d}}_{eg}^i}{-2\hbar\Delta} - \hat{\alpha}^{(0)} \delta_{ij}.$$

(d) **Extra Credit: 5 points**

For the particular case of $|g; J_g = 1/2\rangle \rightarrow |e; J_e = 3/2\rangle$, show that the rank-2 tensor part *vanishes*. Show that the light-shift operator can be written in a basis independent *form* of a scalar interaction (independent of the sublevel), plus an effective Zeeman interaction for a fictitious B-field interacting with the spin 1/2 ground state,

$$\hat{V}_{LS} = V_0(\mathbf{x})\hat{1} + \mathbf{B}_{fict}(\mathbf{x}) \cdot \hat{\boldsymbol{\sigma}}$$

where

$$V_0(\mathbf{x}) = \frac{2}{3}V_1|\vec{\epsilon}_L(\mathbf{x})|^2 \quad (\text{proportional to field intensity}) \text{ and}$$

$$\mathbf{B}_{fict}(\mathbf{x}) = \frac{1}{3}V_1\left(\frac{\vec{\epsilon}_L^*(\mathbf{x}) \times \vec{\epsilon}_L(\mathbf{x})}{i}\right), \quad (\text{proportional to the field ellipticity}),$$

and I have written $E(\mathbf{x}) = E_1\vec{\epsilon}_L(\mathbf{x})$. Use this form to **explain** your results from part (b) on the transition $|g; J_g = 1/2\rangle \rightarrow |e; J_e = 3/2\rangle$.