Problem 1

Spontaneous decay rate on a dipole transition $|\psi_e\rangle \to |\psi_g\rangle$ summed over all possible final states

$$\Gamma = \frac{4}{3\hbar} k^3 \sum_g |\langle \psi_g | \hat{\vec{d}} | \psi_e \rangle|^2$$

Hydrogen, including fine-structure $(nLJM_J) \to \sum_{M_J'} (n'L'J'M_J')$

$$\Gamma = \frac{4}{3\hbar} k^3 \sum_{M_J'} |\langle n'L'J'M_J' | \hat{\vec{d}} | nLJM_J \rangle|^2$$

(a) Expand in spherical basis $\hat{\vec{d}} = \sum_q \vec{e}_q^{\,*} \hat{d}_q$ and use W.E.T

$$\Gamma = \frac{4}{3\hbar} k^3 |\langle n'L'J' \| d \| nLJ \rangle|^2 \underbrace{\sum_{q M_J} |\langle J'M_J' | 1 q \, J M_J \rangle|^2}_{= 1 \text{ by normalization}}$$

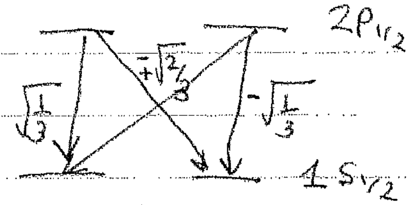
$$\Rightarrow \boxed{\Gamma = \frac{4}{3\hbar} k^3 |\langle n'L'J' \| d \| nLJ \rangle|^2 \quad \text{independent of } M_J}$$

This make sense physically. The vacuum is isotropic and will not care what direction the angular momentum of the atom is pointing in.

Aside: Note that $\Gamma \propto \omega^3$. Thus high frequency transitions decay much more rapidly than low frequency transitions. This make sense from classical electromagnetic theory: The Larmor power (rate of energy rated from an oscillating dipole) goes as ω^3.

(b) Life of of $2p_{1/2}$ in Hydrogen

Allowed transitions



Note branching ratios for decay:

$\frac{1}{3} + \frac{2}{3} = 1$ ✓

(Note Lamb shift puts $2S_{1/2}$ above $2P_{1/2}$)

From part (a), we need only consider one initial M_J since Γ is the same for all

$$\Rightarrow \quad \Gamma(2p_{1/2}) = \frac{4k^3}{3\hbar} \left| \langle 1S_{1/2} \| d \| 2P_{1/2} \rangle \right|^2$$

To calculate the reduced matrix element, pick some allowed transition and use the W.E.T.

$$\underbrace{\langle 1s_{1/2}, M_J = 1/2 | \hat{d}_z | 2P_{1/2}, M_J = 1/2 \rangle}_{} = \langle 1s_{1/2} \| d \| 2p_{1/2} \rangle \underbrace{\langle \tfrac{1}{2} \tfrac{1}{2} | 1 0 \tfrac{1}{2} \tfrac{1}{2} \rangle}_{= \frac{-1}{\sqrt{3}}}$$

Must uncouple spin + orbit ang. mom.

$$|2P_{1/2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}} |2p, 0\rangle |\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}} |2p, 1\rangle |-\tfrac{1}{2}\rangle \quad \text{(From P.S. \#1 prob 3)}$$

$$|1S_{1/2}, \tfrac{1}{2}\rangle = |1s, 0\rangle |\tfrac{1}{2}\rangle$$

$$\Rightarrow \langle 1s_{1/2}, \tfrac{1}{2} | \hat{d}_z | 2p_{1/2}, \tfrac{1}{2} \rangle = \frac{1}{\sqrt{3}} \langle 1s, 0 | \hat{d}_z | 2p, 0 \rangle = \frac{-e}{\sqrt{3}} \langle 1s, 0 | \hat{z} | 2p, 0 \rangle$$

Aside: $\langle 1s, 0 | \hat{z} | 2p, 0 \rangle = \int d^3x \, \psi^*_{1s,0}(\vec{x}) \, z \, \psi_{2p,0}(\vec{x})$

$$z = \sqrt{\frac{4\pi}{3}} \, r \, Y^1_0(\theta, \phi)$$

$$\Rightarrow \langle 1s, 0 | \hat{z} | 2p, 0 \rangle = \sqrt{\frac{4\pi}{3}} \int_0^\infty dr \, u_{10}(r) \, r \, u_{20}(r) \int d\Omega \, Y^0_0 Y^1_0 Y^1_0$$

Aside continued:

$$\int_0^\infty dr\ r\ u_{10}(r)\, u_{21}(r) = \frac{a_0}{\sqrt{6}} \int_0^\infty \bar{r}^4 e^{-\frac{3}{2}\bar{r}}\, d\bar{r} = 1.29\, a_0$$

$$\int d\Omega\ Y_0^0\, Y_0^1\, Y_0^1 = \frac{1}{\sqrt{4\pi}} \underbrace{\int d\Omega\ Y_0^1\, Y_0^1}_{=1} = \frac{1}{\sqrt{4\pi}}$$

$$\Rightarrow \langle 1s_{1/2}\ \tfrac{1}{2} | \hat{d}_z | 2p_{1/2}\ \tfrac{1}{2} \rangle = \left(-\frac{e}{\sqrt{3}}\right)\left(\sqrt{\frac{4\pi}{3}}\right)(1.29\, a_0)\left(\frac{1}{\sqrt{4\pi}}\right)$$

$$= -0.43\ e a_0$$

$$\Rightarrow \langle 1s_{1/2} \| d \| 2p_{1/2} \rangle = \sqrt{3}\ 0.43\ e a_0 = 0.74\ e a_0$$

Thus, $\Gamma(2p_{1/2}) = (0.74)^2 \frac{4}{3\hbar} k^3 (e a_0)^2$

Aside: In atomic units: $\dfrac{\hbar\Gamma}{e^2/a_0} = \frac{4}{3}(k a_0)^3 \left(\dfrac{\langle d\rangle}{e a_0}\right)^2$

$$k = \frac{\omega}{c} = \frac{\hbar\omega}{\hbar c} = \varepsilon \frac{E_0}{\hbar c} = \varepsilon \frac{e^2}{\hbar c\, a_0} = \alpha\varepsilon \frac{1}{a_0}$$

$$\Rightarrow \hbar\Gamma/(e^2/a_0) = \varepsilon^3 \frac{4}{3} \alpha^3 \left(\frac{\langle d\rangle}{e a_0}\right)^2$$

Here $\dfrac{\langle d\rangle}{e a_0} = 0.74$ $\qquad \varepsilon = \dfrac{\hbar\omega}{E_0} = \dfrac{\frac{3}{8}E_0}{E_0} = \dfrac{3}{8}$

$\alpha = \dfrac{1}{137}$ $\qquad$ (Photon for $n=2 \to n=1$)

$$\Rightarrow \hbar\Gamma/(e^2/a_0) = 1.5\times 10^{-8} \qquad \frac{e^2}{a_0} = 27.2\,\text{eV} = 4.1\times 10^{16}\, s^{-1}$$

$$\Rightarrow \Gamma = 6.2\times 10^{8}\, s^{-1}$$

$$\Rightarrow \text{Lifetime}\ \boxed{\tau = 1/\Gamma = 1.6\ ns}$$

Problem Light-Shift (multi-level atoms)

Arbitrary, vector monochromatic field $\vec{E}(\vec{x},t) = Re(\vec{E}(\vec{x})e^{-i\omega_L t})$

driving ground $\leftrightarrow$ excited manifolds

$$\{|g; J_g\rangle \longleftrightarrow |e; J_e\rangle\}$$

Light-shift operator on ground manifold

$$\hat{V}_{LS}(\vec{x}) = -\frac{1}{4}\vec{E}^*(\vec{x}) \cdot \hat{\overleftrightarrow{\alpha}} \cdot \vec{E}(\vec{x})$$

where $\hat{\overleftrightarrow{\alpha}} = -\frac{\hat{\vec{d}}_{ge}\hat{\vec{d}}_{eg}}{\hbar\Delta}$ $\hat{\vec{d}}_{eg} = \hat{\mathbb{P}}_e \hat{\vec{d}} \hat{\mathbb{P}}_g$

polarizability tensor projectors

$\hat{\vec{d}}_{ge} = \hat{\vec{d}}_{eg}^{\dagger}$

(a) Explicit representation in basis of mag-sublevels:

$$\hat{\vec{d}}_{eg} = \hat{\mathbb{P}}_e \hat{\vec{d}} \hat{\mathbb{P}}_g = \sum_{M_e=-J_e}^{J_e} \sum_{M_g=-J_g}^{J_g} |e; J_e M_e\rangle\langle e; J_e M_e| \hat{\vec{d}} |g; J_g M_g\rangle\langle g; J_g M_g|$$

Spherical basis expansion: $\hat{\vec{d}} = \sum_q (-1)^q \vec{e}_{-q} \hat{d}_q = \sum_q \vec{e}_q^{\,*} \hat{d}_q$

$$\Rightarrow \hat{\vec{d}}_{eg} = \sum_{M_e, M_g} \vec{e}_q^{\,*} \langle e; J_e M_e | \hat{d}_q | g; J_g M_g\rangle \, |e; J_e M_e\rangle\langle g; J_g M_g|$$

Aside: W. E. T.

$$\langle e; J_e M_e | \hat{d}_q | g; J_g M_g \rangle = \langle e; J_e \| d \| g; J_g \rangle \langle J_e M_e | 1\, q\, J_g\, M_g \rangle$$

Selection rule $M_e = M_g + q$

$$\Rightarrow \hat{\vec{d}}_{eg} = \sum_{M_g, q} \vec{e}_q^{\,*}\, C_{M_g}^{M_g+q} \, |e; J_e\, M_g+q\rangle\langle g; J_g\, M_g| \, \langle e\|d\|g\rangle$$

Where I have used a short hand for the dipole C-G coef: $C_{m_g}^{m_g+q} = \langle J_e\, m_g+q | 1\, q\, J_g\, m_g \rangle$

Similarly $\hat{\vec{d}}_{ge} = \hat{d}_{eg}^{\dagger} = \sum_{M_g, q} \vec{e}_q\, C_{M_g}^{M_g+q}\, |g, J_g M_g\rangle\langle e, J_e\, M_g+q| \, \langle e\|d\|g\rangle^*$

$$\Rightarrow \hat{d}_{ge}\, \hat{d}_{eg} = \sum_{M_g, M_g'} \sum_{q, q'} \vec{e}_{q'}\, \vec{e}_q^{\,*}\, C_{M_g'}^{M_g'+q'}\, C_{M_g}^{M_g+q}\, |\langle e\|d\|g\rangle|^2$$

<u>Note</u>: Two sums must have two sets of indicies

$$|g, J_g M_g'\rangle \underbrace{\langle e; J_e\, M_g'+q' | e; J_e\, M_g+q\rangle}_{\delta_{M_g'+q',\, M_g+q}} \langle g\, J_g\, M_g|$$

$\Rightarrow$ Selection rule: $M_g' - M_g = q - q'$ (conservation of angular momentum)

$$\delta_{M_g' = M_g + q - q'}$$

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Thus,

$$\hat{d}_{ge}\,\hat{d}_{eg} = |\langle e\|d\|g\rangle|^2 \sum_{M_g} \sum_{q q'} \vec{e}_{q'}\, \vec{e}_q^{\,*}\, C^{M_g+q}_{M_g+q-q'}\, C^{M_g+q}_{M_g}\, |g; J_g, M_g+q-q'\rangle\langle J_g M_g|$$

Putting this together, the polarizability tensor in a spherical basis representation:

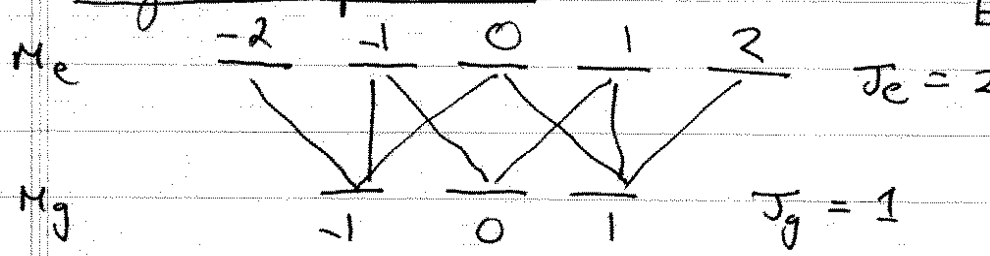
$$\boxed{\hat{\alpha} = \tilde{\alpha}\Big(\sum_{M_g, q} |C^{M_g+q}_{M_g}|^2\, \vec{e}_q\, |g; J_g M_g\rangle\langle g; J_g M_g|\, \vec{e}_q^{\,*} + \sum_{M_g, q\neq q'} C^{M_g+q}_{M_g+q-q'}\, C^{M_g+q}_{M_g}\, \vec{e}_{q'}\, |g; J_g M_g+q-q'\rangle\langle J_g M_g|\, \vec{e}_q^{\,*}\Big)}$$

Here I explicitly write diagonal + off-diag terms

where $\tilde{\alpha} = -\dfrac{|\langle e; J_e\|d\|g; J_g\rangle|^2}{\hbar\Delta}$ (reduced matrix element)

Physical picture:

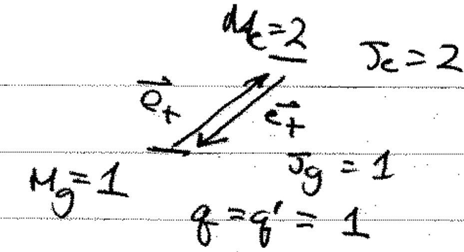
Example



The shift can be thought of as resulting from virtual absorption and emission of photons. If the atom absorbs and reëmits a photon of helicity q $\Rightarrow$ comes back to the same state $\Rightarrow$ Oscillator strength $M_g \leftrightarrow M_e$.

If the atom absorbs photon helicity q and emits q', the deficit goes into atomic ang. mo... $M_g \to M = M_g + q \to M_g = M_e - q'$

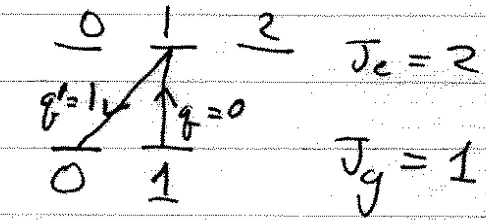
ex: Diagonal term



strength $|C_{M_g=1}^{M_e=2}|^2$

example: Off-diagonal term

Strength $\left(C_0^1 \, C_1^1\right)$



(b) Plane polarized case: $\vec{E}(\vec{x}) = E_1 \, \vec{\varepsilon}_L \, e^{i\vec{k}\cdot\vec{x}}$

$$\Rightarrow \hat{V}_{LS} = -\frac{1}{4}|E_1|^2 \, \vec{\varepsilon}_L^{\,*} \cdot \hat{\overleftrightarrow{\alpha}} \cdot \vec{\varepsilon}_L$$

Case (i): Linear polarization along z $\quad \vec{\varepsilon}_L = \vec{e}_0$

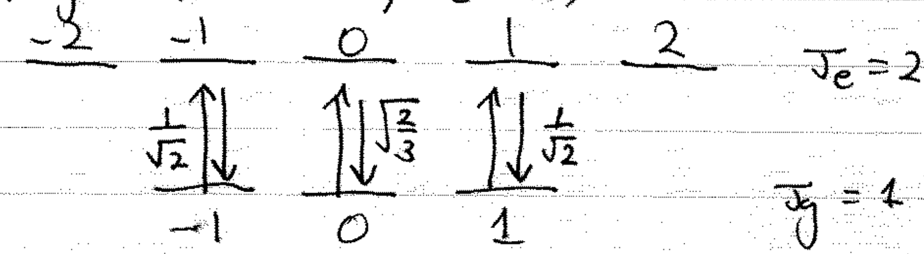
$$\Rightarrow \hat{V}_{LS} = -\frac{1}{4}|E_1|^2 \, \vec{e}_0 \cdot \hat{\overleftrightarrow{\alpha}} \cdot \vec{e}_0$$

$$= \underbrace{-\frac{\tilde{\alpha}}{4}|E_1|^2}_{V_1} \sum_{M_g} |C_{M_g}^{M_g}|^2 \, |g; J_g M_g\rangle\langle g; J_g M_g|$$

$\Rightarrow$ Only diagonal terms (see picture above)

Case $|g; J_g = 1\rangle \longrightarrow |e; J_e = 2\rangle$

Only π-polarization



Clebsch-Gordan coeff $C_{M_g}^{M_g} = \langle 2\, M_g | 1 0 \; 1 M_g \rangle$

$\Rightarrow$ For $\vec{\mathcal{E}}_L = \vec{e}_z$ $\quad |J_g = 1\rangle \to |J_e = 2\rangle$

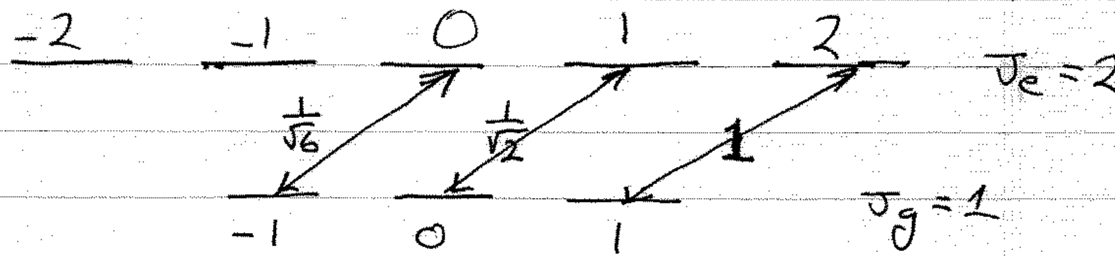
Eigenvectors: Magnetic sublevels $|g, J_g$

Eigenvalues: $\frac{1}{2} V_1$, $\frac{2}{3} V_1$

Sketch: $\Delta < 0 \Rightarrow V_1 < 0$

M_g $\quad -1 \quad 0 \quad 1$ $\quad$ Shifted levels

Case $\vec{\mathcal{E}}_L = \vec{e}_+$ $\quad |J_g = 1\rangle \to |J_e = 2\rangle$



Eigenvectors: $|J_g, M_g\rangle$

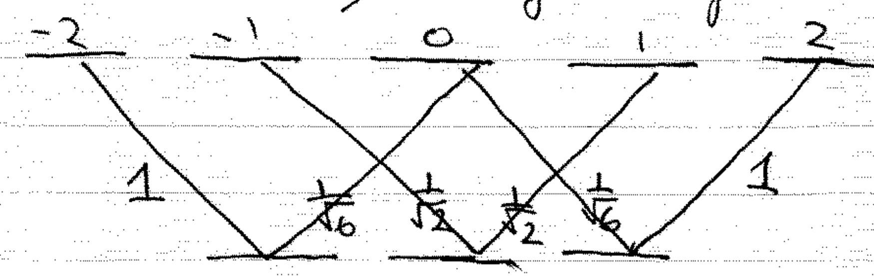
Eigenvalues: $M_g =$ [redacted] : $\frac{1}{6} V_1$, $M_g = 0 : \frac{V_1}{2}$, $M_g = +1 : V_1$

M_g $\quad -1 \quad 0 \quad 1$ $\quad$ ($V_1 < 0$ here)

Case $\vec{\mathcal{E}}_L = \vec{e}_x$ $\quad |J_g = 1\rangle \to |J_e = 2\rangle$

$\vec{e}_x = \frac{-\vec{e}_+ + \vec{e}_-}{\sqrt{2}}$; Not one of spherical basis

$\Rightarrow$ Not just diagonal elements



Light Shift operator for $\vec{\varepsilon}_L = \vec{e}_x = \frac{1}{\sqrt{2}}(-\vec{e}_+ + \vec{e}_-)$

$$\hat{V}_L(\vec{x}) = V_1 \Bigg[\sum_{M_g, q} |C_{M_g}^{M_g+q}|^2 (\vec{e}_x \cdot \vec{e}_q) |J_g M_g\rangle\langle J_g M_g| (\vec{e}_q^{\,*} \cdot \vec{e}_x)$$

$$+ \sum_{M_g, q, q'} C_{M_g+q-q'}^{M_g+q} C_{M_g}^{M_g+q} (\vec{e}_x \cdot \vec{e}_{q'}) |J_g, M_g+q-q'\rangle\langle J_g M_g| \vec{e}_q$$

$$= \frac{V_1}{2} \Bigg[\sum_{M_g} \left(|C_{M_g}^{M_g+1}|^2 + |C_{M_g}^{M_g-1}|^2 \right) |J_g M_g\rangle\langle J_g M_g|$$

$$- \sum_{M_g} \left(C_{M_g+2}^{M_g+1} C_{M_g}^{M_g+1} |J_g, M_g+2\rangle\langle J_g, M_g| + h.c. \right) \Bigg]$$

$$= \frac{V_1}{2} \left(|0\rangle\langle 0| + \frac{1}{12} \left(7|1\rangle\langle 1| + 7|-1\rangle\langle -1| - |1\rangle\langle -1| - |-1\rangle\langle 1| \right) \right)$$

here I have simplified notation $|M_g\rangle \equiv |J_g, M_g\rangle$

Eigenvalues are eigenvectors

In the basis $\{|0\rangle, |1\rangle, |-1\rangle\}$

$$\hat{V}_L \doteq \frac{V_1}{2} \left[\begin{array}{c|cc} 1 & 0 & \\ \hline 0 & \frac{7}{12} & -\frac{1}{12} \\ & -\frac{1}{12} & \frac{7}{12} \end{array} \right]$$

Block diagonal.
We must diagonize the 2x2 matrix

$$\begin{bmatrix} \frac{7}{12} & -\frac{1}{12} \\ -\frac{1}{12} & \frac{7}{12} \end{bmatrix} = \frac{7}{12} \mathbb{1} - \frac{1}{12} \sigma_x \Rightarrow \text{eigenvalues } \frac{7}{12} \mp \frac{1}{12} = \frac{6}{12}, \frac{8}{12}$$

eigenvectors $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{\varepsilon}_L = \vec{e}_x$
$J_g = 1\rangle \to |J_e = 2\rangle$ $\Rightarrow$

$$\boxed{\begin{array}{l} \text{Eigenvector / eigenvalue} \\ |0\rangle : \frac{V_1}{2} \qquad \frac{|1\rangle + |-1\rangle}{\sqrt{2}} : \frac{V_1}{2} \qquad \frac{|1\rangle - |-1\rangle}{\sqrt{2}} : \frac{2}{3} V_1 \end{array}}$$

Thus we see that the eigenvalues for $\vec{\varepsilon}_L = \vec{e}_z$ and $\vec{\varepsilon}_L = \vec{e}_x$ are equal. this is as it must be. What direction we call "x" or "y" or "z" is irrelevant. the choice of "quantization axis" is arbitrary.

What about the eigenvectors for these two cases? For $\vec{\varepsilon}_L = \vec{e}_z$ with z-quantization we found eigenvectors $|M_z = 0\rangle$, $|M_z = \pm 1\rangle$ with eigenvalues

$$\hat{V}_L = \tfrac{2}{3}V_1, \quad \tfrac{1}{2}V_1 \text{ (doubly degenerate)}$$

Thus for $\vec{\varepsilon}_L = \vec{e}_x$ with x-quantization we must have eigenvectors $|M_x = 0\rangle$, $|M_x = \pm 1\rangle$ with eigenvalues

$$\hat{V}_L = \tfrac{2}{3}V_1 \quad \tfrac{1}{2}V_1 \text{ (doubly degenerate)}$$

Now $|M_x\rangle = \hat{D}|M_z\rangle$ where $\hat{D}$ is a rotation matrix

For $J_g = 1$ we can explicitly calculate the rotation matrix, or use symmetry arguments via the spherical basis (see Phys 521, P.S. #6)

$$|1, M_x = 0\rangle = \frac{-1}{\sqrt{2}}\left(|1, M_z = +1\rangle - |1, M_z = -1\rangle\right)$$

$$|1, M_x = \pm 1\rangle = \frac{-i}{\sqrt{2}}\left[\pm|1, M_z = 0\rangle + \frac{1}{\sqrt{2}}\left(|1, M_z = 1\rangle + |1, M_z = -1\rangle\right)\right]$$

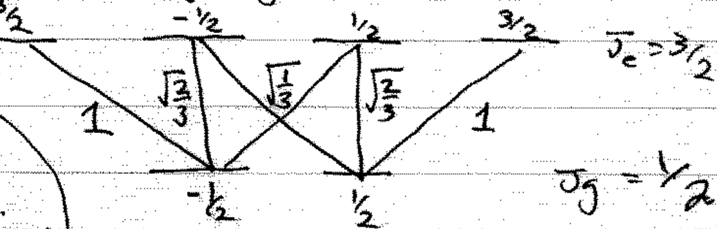
thus

$$\hat{V}_L |M_x = 0\rangle = \tfrac{2}{3}V_1 |M_x = 0\rangle$$

$$\hat{V}_L |M_x = \pm 1\rangle = \tfrac{1}{2}V_1 |M_x = \pm 1\rangle$$

So it all makes sense

Repeat for $|g, J_g = 1/2\rangle \rightarrow |e, J_e = 3/2\rangle$



(C-G coeff's for dipole transitions)

<u>Case (i)</u> $\vec{\mathcal{E}}_L = \vec{e}_z$

$$\boxed{\begin{array}{ll}\text{Eigenvector} & |J_g, M_g = \pm 1/2\rangle \\ \text{Eigenvalues} & \frac{2}{3}V_1 \quad \text{(doubly degenerate)}\end{array}}$$

<u>Case (ii)</u> $\vec{\mathcal{E}}_L = \vec{e}_+$

$$\boxed{\begin{array}{ll}\text{Eigenvectors} & |J_g, M_g = \pm \tfrac{1}{2}\rangle \\ \text{Eigenvalues} & |M_g = 1/2\rangle : \ V_1 , \quad |M_g = -1/2\rangle : \ \frac{1}{3}V_1\end{array}}$$

<u>Case (iii)</u> $\vec{\mathcal{E}}_L = \vec{e}_x = (-\vec{e}_+ + \vec{e}_-)/\sqrt{2}$

Unlike the $J_g = 1$ case, the light-shift operator is <u>diagonal</u> here since there are no $\Delta M_g = \pm 2$ coherences possible in the ground state

$$\hat{V}_{LS} = \frac{V_1}{2} \sum_{M_g} \left(|C_{M_g}^{M_g+1}|^2 + |C_{M_g}^{M_g-1}|^2 \right) |J_g M_g\rangle\langle J_g M_g|$$

$$= \frac{V_1}{2} \left[\left(1 + \frac{1}{3}\right) |\tfrac{1}{2}\rangle\langle\tfrac{1}{2}| + \left(\frac{1}{3} + 1\right) |-\tfrac{1}{2}\rangle\langle -\tfrac{1}{2}| \right)$$

$$= \frac{2}{3} V_1 \left(|\tfrac{1}{2}\rangle\langle\tfrac{1}{2}| + |-\tfrac{1}{2}\rangle\langle -\tfrac{1}{2}| \right)$$

$\Rightarrow$ Eigenvectors $|J_g, M_g = \pm 1/2\rangle$ — Same as $\vec{\mathcal{E}}_L = \vec{e}_z$

Eigenvalues $\frac{2}{3} V_1$ (doubly ...) — as it must be

(c) The polarizability tensor can be written in terms of irreducible tensors

Let $\hat{T}_{ij} = d^i_{ge}\, d^j_{eg}$ outer product of two vectors

As discussed in class, any such Cartesian tensor can be expanded in terms of irreducible tensors

$$\hat{T}_{ij} = \hat{T}^{(0)}_{ij} + \hat{T}^{(1)}_{ij} + \hat{T}^{(2)}_{ij}$$

where $\hat{T}^{(0)}_{ij} = \text{Trace}(\hat{T}_{ij})\,\frac{\delta_{ij}}{3} = \frac{1}{3}\,\vec{d}_{ge}\cdot\vec{d}_{eg}$

Antisymmetric: $\hat{T}^{(1)}_{ij} = \frac{\hat{d}^i_{ge}\hat{d}^j_{eg} - \hat{d}^j_{ge}\hat{d}^i_{eg}}{2} = \frac{1}{2}\epsilon_{ijk}\left(\hat{\vec{d}}_{ge}\times\hat{\vec{d}}_{eg}\right)_k$

Symmetric, Traceless: $\hat{T}^{(2)}_{ij} = \frac{\hat{d}^i_{ge}\hat{d}^j_{eg} + \hat{d}^j_{ge}\hat{d}^i_{eg}}{2} - \frac{1}{3}\hat{\vec{d}}_{ge}\cdot\hat{\vec{d}}_{eg}$

Thus with $\hat{\alpha}_{ij} = -\frac{1}{\hbar\Delta}\hat{T}_{ij}$

we arrive at the desired expansion

$$\hat{V}_{LS} = -\frac{1}{4}\vec{E}^*(\vec{x})\cdot\hat{\alpha}\,\vec{E}(\vec{x}) = -\frac{1}{4}E^*_i E_j\,\hat{\alpha}_{ij}$$

$$= -\frac{1}{4}\left(\hat{\alpha}^{(0)}|\vec{E}(\vec{x})|^2 + \hat{\alpha}^{(1)}\cdot(\vec{E}^*\times\vec{E}) + \vec{E}^*\cdot\hat{\alpha}^{(2)}\cdot\vec{E}\right)$$

$$\hat{\alpha}^{(K)}_{ij} = -\frac{1}{\hbar\Delta}\hat{T}^{(K)}_{ij}$$

(d) For the particular case $|g; J_g = 1/2\rangle \to |e; J_e = 3/2\rangle$

$\hat{V}_{LS}$ acts on the ground manifold $\{|J_g, 1/2\rangle, |J_g, -1/2\rangle\}$

The rank-2 part $\hat{V}_{LS}^{(2)} = -\frac{1}{4} E_i^* E_j \hat{\alpha}_{ij}^{(2)}$

has matrix elements $\langle J_g = 1/2, M_g' | \hat{V}_{LS}^{(2)} | J_g = 1/2, M_g\rangle$

$$= \langle \tfrac{1}{2} \| \hat{V}_{LS}^{(2)} \| \tfrac{1}{2}\rangle \underbrace{\langle \tfrac{1}{2} M_g' | 2 q \; \tfrac{1}{2} M_g\rangle}$$

this C-G coeff vanishes by the triangle inequality

Thus,
$$\hat{V}_{LS} = -\frac{1}{4} |\vec{E}_0|^2 \hat{\alpha}^{(0)} - \frac{1}{4} (\vec{E}^* \times \vec{E}) \cdot \hat{\alpha}^{(1)}$$

$\downarrow$ Scalar $\qquad$ $\downarrow$ Vector

Acting on spin-1/2 the scalar part must be proportional to the identity and the vector part to $\hat{\vec{\sigma}}$ since any operator acting on the 2D ~~Hilbert~~ Hilbert space is of this form

Thus,
$$\boxed{\begin{aligned} \hat{V}_{LS} &= V_0(\vec{x})\, \hat{\mathbb{1}} + \vec{B}_{eff}(\vec{x}) \cdot \hat{\vec{\sigma}} \\ \text{Where } V_0(\vec{x}) &= \tfrac{1}{2} \mathrm{Tr}(\hat{V}_{LS}) \\ \vec{B}_{eff} &= \tfrac{1}{2} \mathrm{Tr}(\hat{V}_{LS}\, \hat{\vec{\sigma}}) \end{aligned}}$$