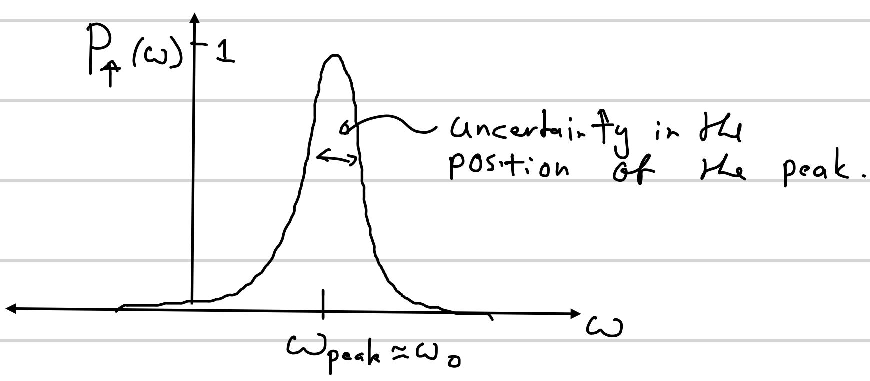
Physics 522
Problem Set #12 Solutions

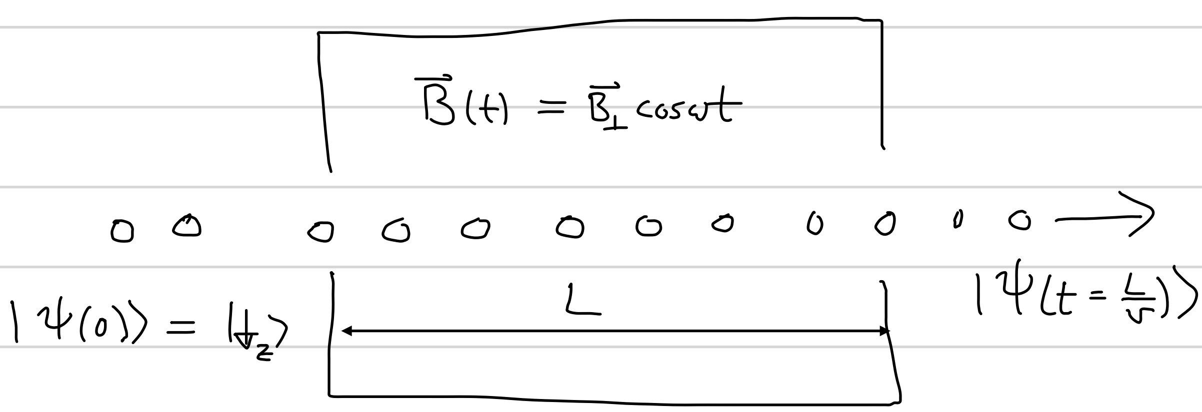
Problem 1: Magnetic Resonance: Rabi vs. Ramsey

A key problem in precision spectroscopy to to tune an oscillator to an absorption two-level resonance. We measure the different between ω + ω_0 by measuring the probability to transfer the (pseudo) spin from $|\downarrow_z\rangle \Rightarrow |\uparrow_z\rangle$ as a function of the detuning. The precision with which we can measure ω_0 depends sharpness of the resulting curve. If the curve $P_\uparrow(\omega)$ is very sharp, then we have small uncertainty the $\omega=\omega_0$ and the peak. If it is broad, then we can be detuned by a substantial amount a still reach high probability

Qualitatively:



(a) In the original Rabi experiment, the spins pass through a cavity we oscillating transverse field $\vec{B}_\perp \cos\omega t$. If the cavity has length L, and the spin moves through the cavity at speed v, the interact for a time $t = L/v$



The evolution of the spin is governed by the spin-resonance Hamiltonian in the RWA

In the rotating frame: $\hat{H} = -\frac{\hbar\Delta}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}\hat{\sigma}_x \Rightarrow$ Rabi evolution: $\hat{U} = e^{-\frac{i}{\hbar}\hat{H}t}$

$$\hat{U}(t) = e^{i\vec{\Omega}_{tot}\cdot\frac{\vec{\sigma}}{2}} = \cos\frac{\Omega_{tot}t}{2}\hat{\mathbb{1}} + i\frac{\Delta}{\Omega_{tot}}\sin\frac{\Omega_{tot}t}{2}\hat{\sigma}_z - i\frac{\Omega}{\Omega_{tot}}\sin\frac{\Omega_{tot}t}{2}\hat{\sigma}_x$$

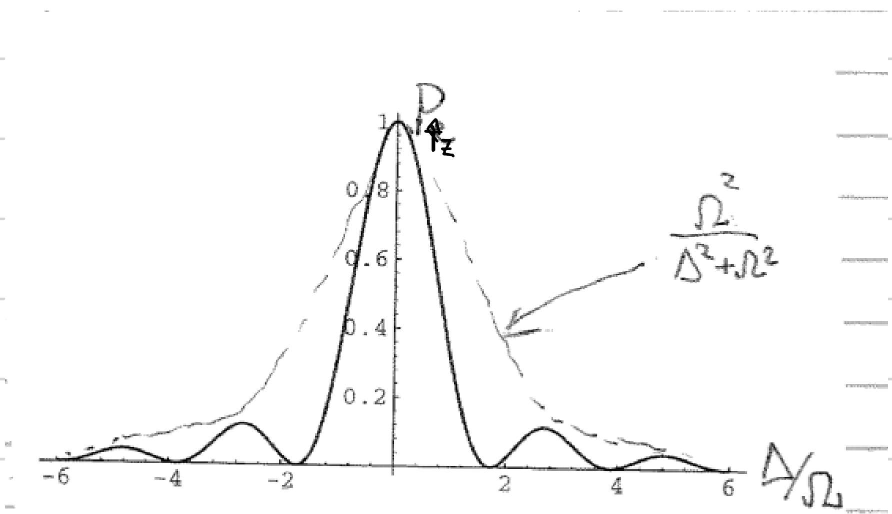
The spins start in $|\downarrow_z\rangle$ and evolve according to the Rabi Hamiltonian for a time $T = \frac{L}{v}$.
The probability to be in the excited $|\uparrow_z\rangle$ state after emerging from the cavity is thus

$$P_{\uparrow_z} = |\langle \uparrow_z | U(t = L/v) | \downarrow_z \rangle|^2 = \frac{\Omega^2}{\Omega_{tot}^2} \sin^2(\Omega_{tot} L/2v)$$

$$= \frac{\Omega^2}{\Omega^2 + \Delta^2} \sin^2\left(\sqrt{\Omega^2 + \Delta^2}\, \frac{L}{2v}\right)$$

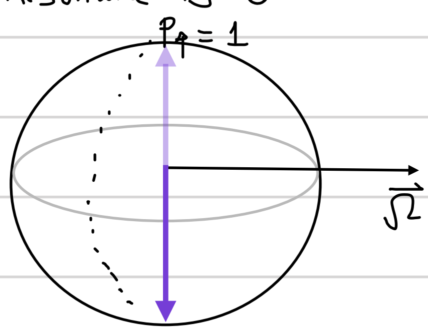
The bare Rabi frequency is chosen so that $\Omega T = \frac{\Omega L}{v} = \pi$ ("π-pulse")

$$\Rightarrow \boxed{P_{\uparrow_z} = \frac{1}{1 + \left(\frac{\Delta}{\Omega}\right)^2} \sin^2\left(\sqrt{1 + \left(\frac{\Delta}{\Omega}\right)^2}\, \frac{\pi}{2}\right)}$$

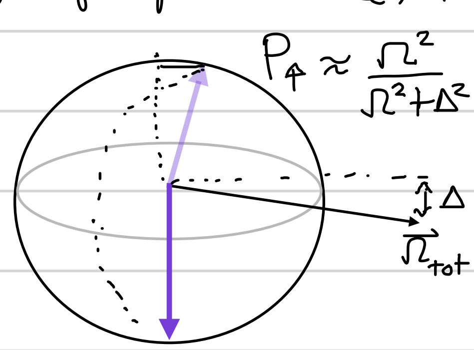


The linewidth of this curve determines the uncertainty of the resonance frequency. For the Rabi method, $\boxed{\delta(\omega - \omega_0) \sim \frac{1}{\Omega} = \frac{T}{\pi}}$. This is just an expression of the Fourier time – frequency uncertainty principle.

Resonane $\Delta = 0$



Slightly of resonance, $|\Delta| << \Omega$



It is clear from these sketches that when Δ is on the order of Ω or larger, the population transfer to the excited state decreases substantially. Since we have $\Omega T = \pi$, we

(e) Quantitatively, we can turn to the adiabatic theorem of quantum mechanics

Now with a changing detuning we have the time dependent Hamiltonian in the rotating frame

$$H_{eff} = -\frac{\hbar}{2}\vec{\Omega}_{eff}(t)\cdot\hat{\vec{\sigma}}$$

where $\vec{\Omega}_{eff}(t) = \Omega\vec{e}_x + \Delta(t)\,\vec{e}_z$

We can ~~instantly~~ write down the instaneous eigenvectors and eigenvalues in a snap:

$$\hat{H}_{eff} = -\frac{\hbar}{2}\tilde{\Omega}(t)\,\hat{\sigma}_{n(t)}$$

where $\tilde{\Omega}(t) = \sqrt{\Omega^2 + (\Delta(t))^2}$

and $\hat{\sigma}_{n(t)} = \vec{e}\cdot\hat{\vec{\sigma}}$, $\vec{e}_{n(t)} = \cos\theta(t)\,\vec{e}_z + \sin\theta(t)\,\vec{e}_x$

$$\theta(t) = \tan^{-1}\left(\frac{\Omega}{\Delta(t)}\right)$$

$$\cos\theta(t) = \frac{\Delta(t)}{\tilde{\Omega}(t)} \qquad \sin\theta(t) = \frac{\Omega}{\tilde{\Omega}(t)}$$

We thus have the eigenvalues (since $\hat{\sigma}_n = \pm 1$)

$$E_{\pm}(t) = \mp\frac{\hbar}{2}\tilde{\Omega}(t) = \mp\frac{\hbar}{2}\sqrt{\Omega^2 + \Delta(t)^2}$$

Corresponding to eigenvectors

$$|\pm\rangle_{n(t)} = \cos\frac{\theta(t)}{2}|\pm\rangle_z + i\sin\left(\frac{\theta(t)}{2}\right)|\mp\rangle_z$$

$$|+\rangle_z = |e\rangle \qquad |-\rangle_z = |g\rangle$$

are most sensitive to the detuning by making T longer and Ω smaller, maintaining $\Omega T = \pi$.

(b) Now suppose the spins have a distribution of velocities, characteristic of thermal beams

$$f(v) = 2\frac{v^3}{v_0^4} e^{-\frac{v^2}{v_0^2}}, \quad \text{where} \quad v_0 = \sqrt{2k_BT/m}$$

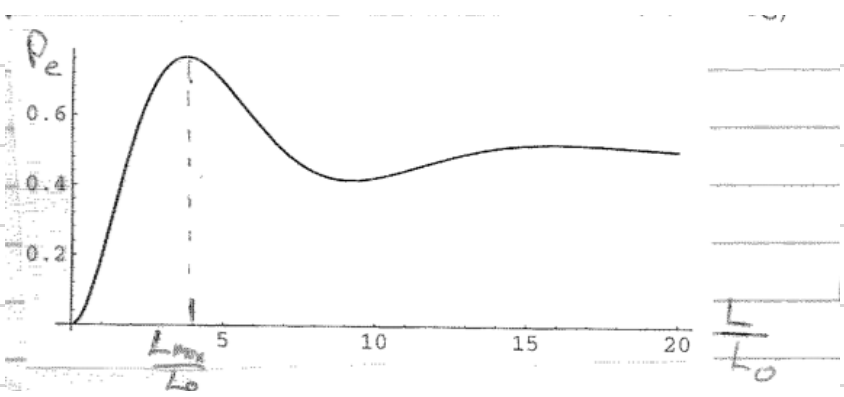
Note: For this distribution:

- Most probable speed $v_p = 1.22 v_0$, • Average speed: $\bar{v} = 1.33 v_0$, • RMS speed $\sqrt{\overline{\Delta v^2}} = 1.42 v_0$.

For a detuning Δ and speed v, we have $P_\uparrow(\Delta, v) = \frac{1}{1+\left(\frac{\Delta}{\Omega}\right)^2} \sin^2\left[\left(1+\frac{\Delta^2}{\Omega^2}\right)^{1/2} \frac{\Omega L}{2v}\right]$.
At zero detuning $P_\uparrow(0, v) = \sin^2\left(\frac{L\Omega}{2v}\right)$. Now we must average over the velocity distribution:

$$P_\uparrow = \int_0^\infty dv \, f(v) \, P_\uparrow(0, v) = \int_0^\infty dx \; f(x) \sin^2\left(\frac{\theta_0}{2x}\right) \quad \text{[expressed in dimensionless variables]}$$

where $x = \frac{v}{v_0}$ (dimensionless velocity), $\theta_0 \equiv \frac{\Omega L}{v_0}$ (Angle of precession on the Bloch sphere for spins traveling @ v)

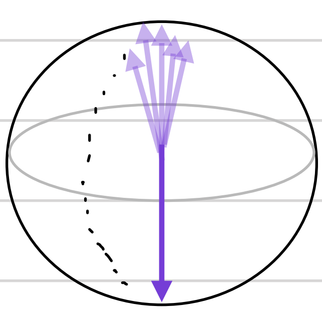


$\left(L_0 \equiv \frac{v_0}{\Omega}\right)$

The maximum $P_\uparrow$ occurs at $L_{max} = 3.77 L_0 \cong 1.20\pi \frac{v_0}{\Omega}$. We can understand this from the fact that the most probable speed is $1.2 v_0$

$$\Rightarrow \frac{\Omega L_{max}}{1.2 v_0} = \pi$$

On the Bloch Sphere



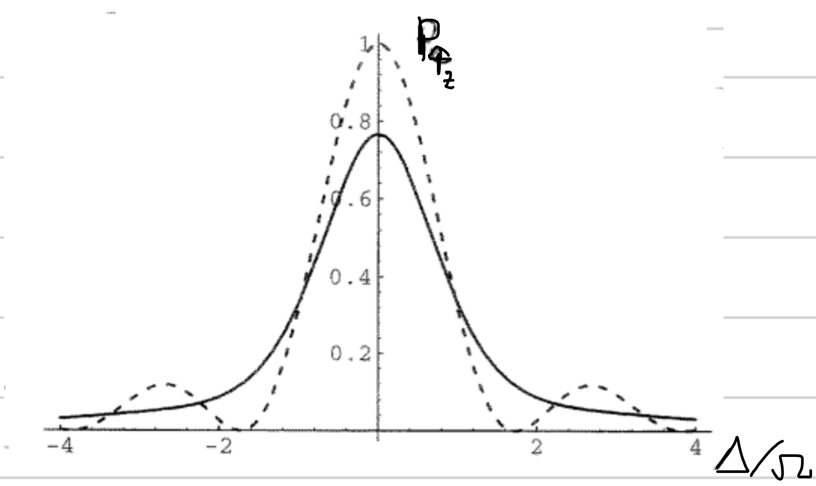
Distribution of speeds → Distribution of interaction times → Distribution of rotation angles on the Bloch sphere.

This is an example of "inhomogeneous broadening." The "damped" Rabi oscillations plotted above are not due to irreversible decay. Rather they are due to the spread in rotation angles due to the spread in velocities.

The total probability to excite the spin as a function of detuning is the the average of $P_{\uparrow}(\Delta, v)$ over the distribution of speeds $f(v)$. We set $L = L_{max}$, so $\Theta_0 = 1.2\pi$

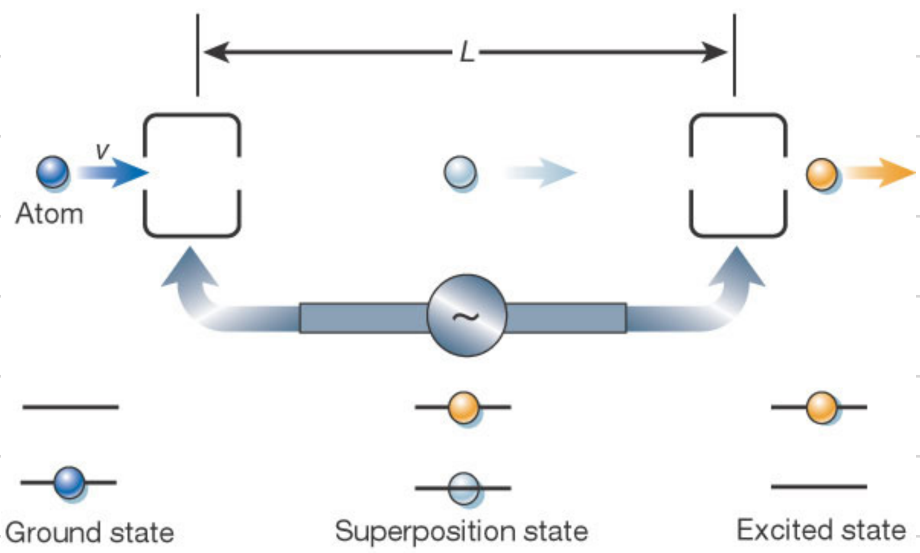
$$P_{\uparrow}(\Delta) = \frac{1}{1+\left(\frac{\Delta}{\Omega}\right)^2} \int_0^{\infty} \sin^2\left[\left(1+\frac{\Delta^2}{\Omega^2}\right)^{1/2} \frac{\Theta_0}{2} \frac{1}{x}\right] \left(2x^3 e^{-x^2}\right) dx$$

Solve numerically. Here the solution is plotted as a function of Δ/Ω with $L = L_{max}$.



For reference, I have added the curve for mono-energetic spins (dashed curve). We see that the spread in speeds leads to a broadening of the resonance lineshape.

(c) We now consider the Ramsey separated zone method.

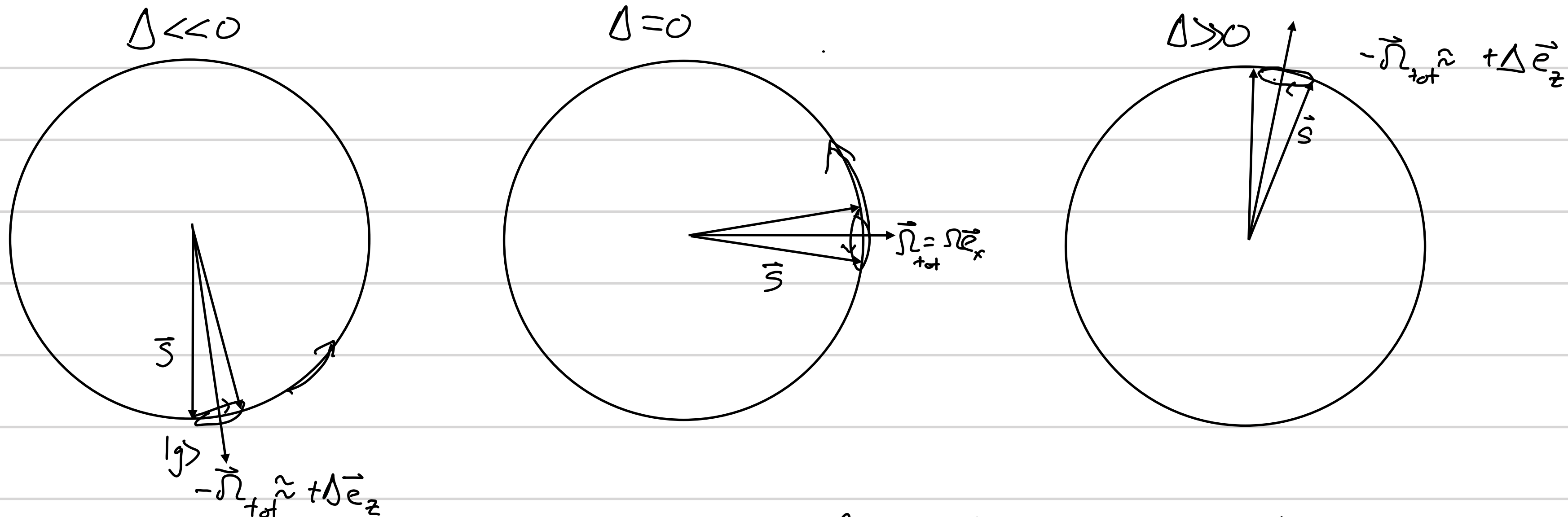


The key idea is to separate two short interaction zones that achieve $\pi/2$ rotations by a long interaction-free zone. As we will show, the precission with which we can measure $\omega - \omega_0$ is set by L, which can be made much longer than for the single zone case.

Problem 2: Adiabatic Passage

The Hamiltonian in the rotating frame $\hat{H} = -\frac{\hbar\Delta}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}\hat{\sigma}_x$.
By sweeping the detuning we can "adiabatically" transfer the system from the ground state to the excited state.

(a) We can get a qualitative picture using the mapping of the two-level atom to a pseudo-spin and the Bloch sphere picture of dynamics,



* I show the negative of the torque vector here

As we sweep the detuning, the direction of the torque vector changes. If the direction of the torque is fixed, the (pseudo) spin precesses around the local torque vector and a frequency $\Omega_{tot} = \sqrt{\Omega^2 + \Delta^2}$. As the direction of the torque changes, the spin will adiabatically follow the direction of the torque if the rate of change of the torque is slow compared to the local precession rate. The slowest precession rate is Ω, thus we expect adiabatic following when

$$\left| \frac{d}{dt} \frac{\vec{\Omega}_{tot}}{|\vec{\Omega}_{tot}|} \right| \ll \Omega, \quad \text{where } \vec{\Omega}_{tot} = -\Delta(t)\vec{e}_z + \Omega\vec{e}_x$$

(b) Quantitatively, ~~we~~ we can turn to the adiabatic theorem of quantum mechanics

Now with a changing detuning we have the time *dependent* Hamiltonian in the rotating frame

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where $\tilde{\Omega}(t) = \sqrt{\Omega^2 + (\Delta(t))^2}$

and $\hat{\sigma}_{n(t)} = \vec{e}\cdot\hat{\vec{\sigma}}$, $\vec{e}_{n(t)} = \cos\theta(t)\,\vec{e}_z + \sin\theta(t)\,\vec{e}_x$,

$$\theta(t) = \tan^{-1}\left(\frac{\Omega}{\Delta(t)}\right)$$

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We thus have the eigenvalues (since $\hat{\sigma}_n = \pm 1$)

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Corresponding to eigenvectors

$$|\pm\rangle_{n(t)} = \cos\frac{\theta(t)}{2}\,|\pm\rangle_z + i\sin\left(\frac{\theta(t)}{2}\right)|\mp\rangle_z$$

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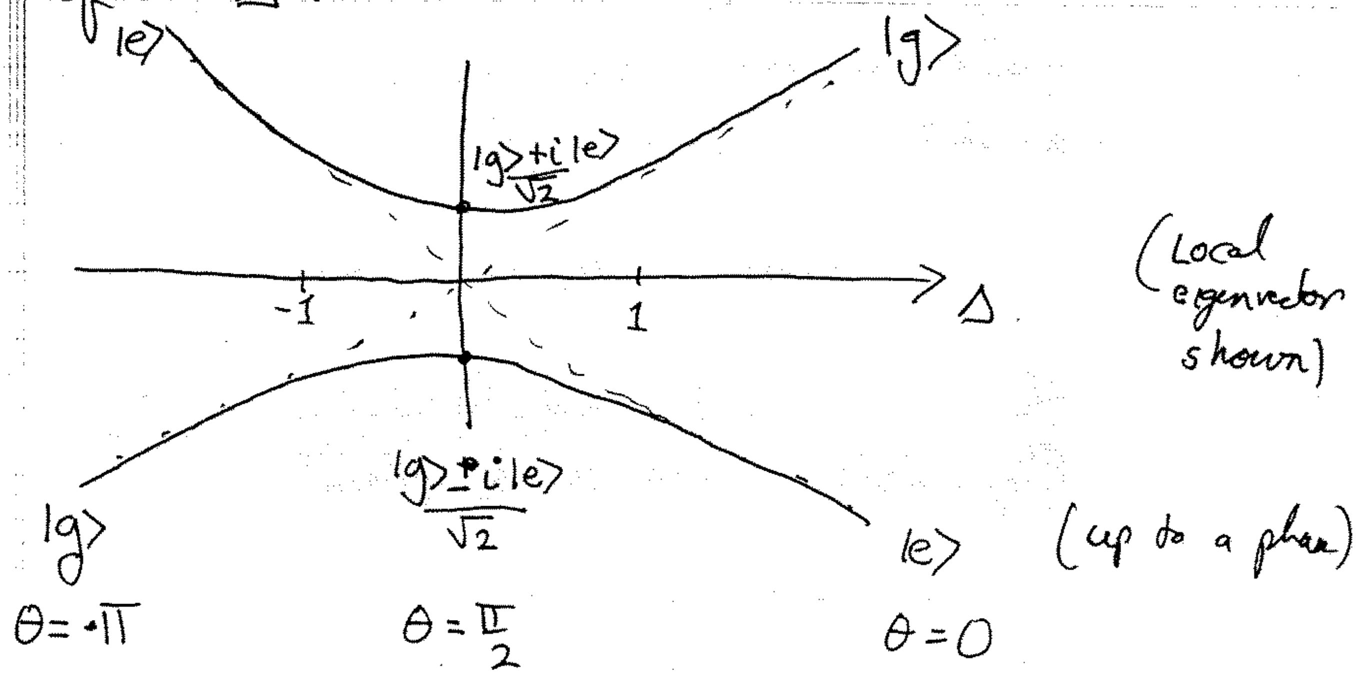
A graph of these eigenvalues as a function of Δ:



According to the adiabatic theorem of quantum mechanics, for a time dependent Hamiltonian that varies slowly, if we start in an eigenstate we stay in the local eigenstate. Thus according to the curve above, we we adiabatically follow the lower branch, so that the state evolve from $|g\rangle \to \frac{1}{\sqrt{2}}(|g\rangle - i|e\rangle) \to |e\rangle$

This requires $\frac{d}{dt}\theta(t) \ll \tilde{\Omega}(t)$ (adiabatic condition

The local eigenstates of H_{eff} are sometimes known as the "dressed states", since the laser field "dresses" the bare atom states

$$|+\rangle_{n(t)} = \frac{1}{\sqrt{2}}\left(\sqrt{1+\frac{\Delta}{\tilde{\Omega}}}\,|e\rangle + i\sqrt{1-\frac{\Delta}{\tilde{\Omega}}}\,|g\rangle\right)$$

The adiabatic theorem of quantum mechanics says that if we have a Hamiltonian which is time dependent, $\hat{H}(t)$, then given a state at $t=0$ which is an eigenstate of $\hat{H}(0)$

i.e. $|\psi(0)\rangle = |u_n(0)\rangle$ where $\hat{H}(0)|u_n(0)\rangle = E_n(0)|u_n(0)\rangle$

The system will <u>adiabatically follow</u> the Eigenstate (up to a phase)

$$|\psi(t)\rangle \Rightarrow |u_n(t)\rangle$$

if $\hat{H}(t)$ changes <u>slowly</u> compared to the frequency associated with energy splittings.

Here the local eigenstate

$$|\psi(t)\rangle = |+\rangle_{n(t)} = \frac{1}{\sqrt{2}}\left(\sqrt{1+\frac{\Delta}{\tilde{\Omega}}}\,|e\rangle + i\sqrt{1-\frac{\Delta}{\tilde{\Omega}}}\,|g\rangle\right)$$

Well below resonance

$\Delta \ll \Omega \qquad |+\rangle_{n(t)} \Rightarrow |g\rangle$

Well above

$\Delta \gg \Omega \qquad |+\rangle_{n(t)} \Rightarrow |e\rangle$

Adiabatic if $\left|\frac{1}{\tilde{\Omega}}\frac{d\tilde{\Omega}}{dt}\right| \ll \Omega$

↑ smallest splitting

Also require rapid compared to Γ