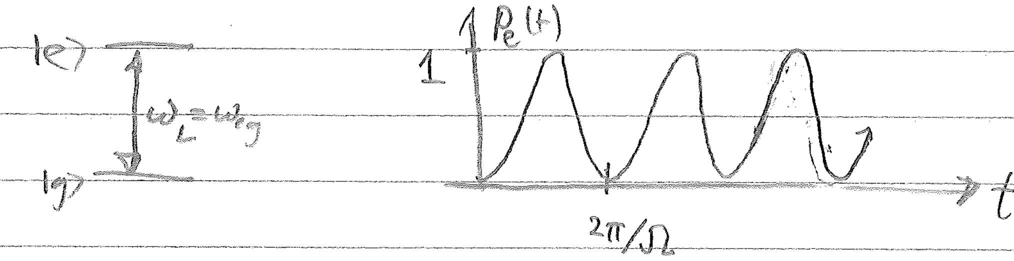
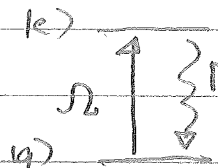
Notes 14: Dissipation and the Master Equation

We have seen that when a monochromatic field is near resonance with a two-level transition coherent Rabi oscillations are driven.



We also know, however, that the excited atom has a finite lifetime due to perturbations from the environment including the fundamental spontaneous emission.



- Coherent drive: Ω = Rabi freq
- Spontaneous decay: Γ = spont. emission

The finite lifetime implies that $|e\rangle$ is no longer a "stationary state", and in the absence of any drive, the probability to stay in the excited state decays. If at $t=0$ we prepare the atom in the $|e\rangle$-state:

$$P_e(t) = e^{-\Gamma t}$$

We can account for this through the probability amplitude: $c_e(t)$, $P_e(t) = |c_e(t)|^2$

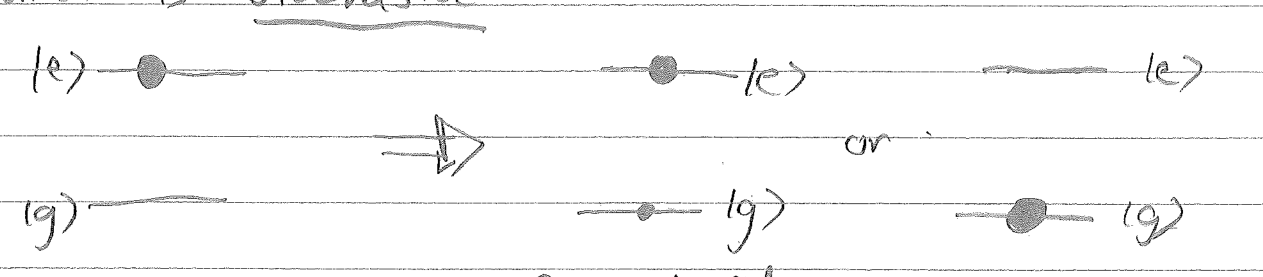
where $$c_e(t) = e^{-\frac{i}{\hbar}E_e t - \frac{\Gamma}{2}t}$$

$$\Rightarrow \quad P_e(t) = |c_e(t)| = e^{-\Gamma_e t}$$

Note: $$c_e(t) = e^{-\frac{i}{\hbar}(E_e - i\frac{\hbar}{2}\Gamma)t}$$

Thus, the effect of the finite lifetime of the excited state is to give its energy eigenvalue an imaginary part. This is rigorous in the theory of "open quantum systems".

What about the ground state? For a two-level system, as population leaves the excited state it must appear in the ground state. What is we set $P_g(t) = 1 - P_e(t) = 1 - e^{-\Gamma t}$.
But there is no probability amplitude that with a complex energy. The reason is that probability amplitude is not returned to the ground state continuously, but as a quantum jump. Thus the evolution is stochastic



Prop: $dp = \Gamma dt$ Prop: $1 - \Gamma dt$

$|\psi\rangle = |e\rangle$ $\quad |\psi\rangle = (1-\Gamma dt)|e\rangle$ and $+\Gamma dt\,|g\rangle$ $\quad |\psi\rangle = |g\rangle$

If we don't collect the photon, then we no longer have a single wave function to describe the state of the atom. Instead the atom is described by a <u>statistical mixture</u> of different wave functions

$$\{ P_j \, , \, |\psi^{(j)}\rangle \}$$

Probability that we describe the state with ket $|\psi^{(j)}\rangle$

<u>Note:</u> the statistical mixture should not be confusing with a <u>coherent superposition</u>. This is very subtle. The state $|\psi^{(j)}\rangle$ is known as a <u>pure state</u>, because we have one wave function description. But, $|\psi^{(j)}\rangle$ can be a superposition, for example, of $|e\rangle$ and $|g\rangle$.

Statistical mixtures are used when we lose information about the state. We now have a more general kind of state which now has a kind of classical uncertainty because we have lost information.

$$|\psi(0)\rangle = |e\rangle \begin{cases} \nearrow \; dp = \Gamma dt & \text{jump} \\ \searrow \; dp = 1 - \Gamma dt & \text{no jump} \end{cases}$$

The Density Matrix

We saw that spontaneous emission leads to decay of the probability amplitude to be in the excited state. This also affects the coherence in a superposition. In other words, spontaneous emission affects the phase relationship between $|e\rangle$ and $|g\rangle$.

For this reason, we specify the state by specifying terms bilinear in the amplitudes.

$$\text{E.g.} \quad c_e c_g^* = \underbrace{|c_e(t)|\,|c_g(t)|}_{\substack{\text{Amplitude}\\ \text{product}}} \; e^{\overbrace{i(\phi_e - \phi_g)}^{\text{Phase difference}}}$$

For our two-level atom, we define a 4x4 matrix known as the "density matrix"

$$\rho_{\alpha\beta} = \overline{c_\alpha c_\beta^*}$$

ensemble average over mixture

$$\hat{\rho} = \begin{bmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{bmatrix}$$

$\rho_{gg} = \overline{|c_g|^2}$ = Population in ground state

$\rho_{ee} = \overline{|c_e|^2}$ = Population in excited state

$$\rho_{ge} = \overline{c_g c_e^*} = \rho_{eg}^* = \text{Coherence}$$

Properties of the density Matrix

- Hermitian: $\hat{\rho}^\dagger = \hat{\rho}$
- Normalization $P_e + P_g = 1 \Rightarrow \rho_{ee} + \rho_{gg} = 1$

$$\Rightarrow \text{Trace}(\hat{\rho}) = 1$$

- Expectation values:

$$\langle \hat{A} \rangle = \sum_j P_j \langle \psi^{(j)} | \hat{A} | \psi^{(j)} \rangle \quad \left(\text{Average of different } |\psi^{(j)}\rangle\right)$$

$$= \overline{\langle \psi^{(j)} | \hat{A} | \psi^{(j)} \rangle}$$

$$= \overline{|c_e|^2} \langle e | \hat{A} | e \rangle + \overline{|c_g|^2} \langle g | \hat{A} | g \rangle$$

$$+ \overline{c_e c_g^*} \langle g | A | e \rangle + \overline{c_g c_e^*} \langle e | A | g \rangle$$

$$= \sum_{\alpha\beta} \rho_{\alpha\beta} A_{\beta\alpha} = \text{Tr}(\hat{\rho} \hat{A})$$

- Operator representation

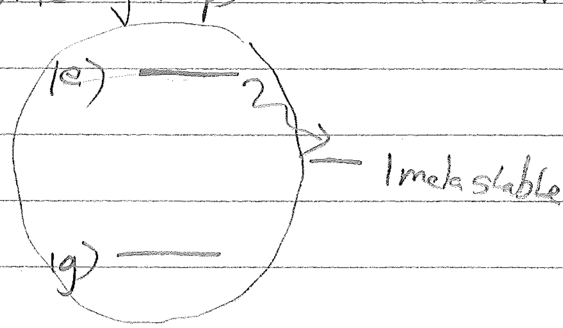
$$\hat{\rho} = \overline{|c_e|^2} |e\rangle\langle e| + \overline{|c_g|^2} |g\rangle\langle g|$$

$$+ \overline{c_e c_g^*} |e\rangle\langle g| + \overline{c_g c_e^*} |g\rangle\langle e|$$

$$\Rightarrow \hat{\rho} = \sum_j P_j |\psi^{(j)}\rangle\langle \psi^{(j)}|$$

Dynamics with dissipation

If the population decays predominantly outside the two-level system then we have no stochastic jumps back to the ground state.



In this case, total probability in $\{|g\rangle, |e\rangle\}$ is not conserved, but we have a set of continuous evolutions for c_e + c_g in the absence of driving

$$\begin{cases} \frac{d}{dt} c_e(t) = \left(\frac{-i}{\hbar} E_e - \frac{\Gamma}{2}\right) c_e \\ \frac{d}{dt} c_g(t) = -\frac{i}{\hbar} E_g c_g \end{cases}$$

The density matrix elements evolve as

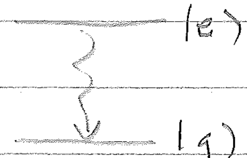
$$\frac{d}{dt} \rho_{ee} = \frac{d}{dt} \overline{c_e c_e^*} = -\Gamma \rho_{ee}$$

$$\frac{d}{dt} \rho_{eg} = \frac{d}{dt} \overline{c_e c_g^*} = -\frac{i}{\hbar}(E_e - E_g) \rho_{eg} - \frac{\Gamma}{2} \rho_{eg}$$

$$\frac{d}{dt} \rho_{gg} = 0$$

Now if we restricted to only two-levels, population in the ground state must be fed from spontaneous emission from the excited state

$$\frac{d}{dt}(\rho_{ee}+\rho_{gg}) = 0 \Rightarrow \frac{d\rho_{gg}}{dt} = -\frac{d}{dt}\rho_{ee}$$



$$\frac{d\rho_{gg}}{dt} = +\Gamma\rho_{ee}$$

In the absense of driving, we thus have the following solution for the decay of the two level atom:

$$\rho_{ee}(t) = e^{-\Gamma t}\rho_{ee}(0) \qquad \rho_{gg}(t) = 1 - e^{-\Gamma t}\rho_{ee}(0)$$

$$= e^{-\Gamma t}\rho_{gg}(0) + (1-e^{-\Gamma t})$$

$$\rho_{eg}(t) = \rho^*_{ge}(t) = \underbrace{e^{-i\omega_{eg}t - \frac{\Gamma}{2}t}}_{\text{oscillating decaying coherence}}$$

Including driving

Excluding spontaneous emission the Schrödinger equation for the driven two level atom, in the RWA (in the rotating frame)

$$\frac{d}{dt}c_e = -i\underbrace{(\omega_{eg}-\omega_L)}_{-\Delta}c_e + i\frac{\Omega}{2}c_g$$

$$\frac{d}{dt}c_g = i\frac{\Omega}{2}c_e$$

Putting in spontaneous emission

$$E_e \Rightarrow E_e - i\frac{\hbar\Gamma}{2}$$

or $\Delta = \omega_L - \left(\frac{E_e - E_g}{\hbar}\right) \Rightarrow \Delta + i\frac{\hbar\Gamma}{2}$

So:

$$\left.\begin{aligned} \frac{d}{dt} c_e &= \left(+i\Delta - \frac{\Gamma}{2}\right) c_e + i\frac{\Omega}{2} c_g \\ \frac{d}{dt} c_g &= i\frac{\Omega}{2} c_e \end{aligned}\right\}$$

Excluding repopulation of $|g\rangle$ by spontaneous jumps

These equations can now be translated into equations of motion for elements of the density operator:

$$\frac{d}{dt}\rho_{ee} = \frac{d}{dt}\left(c_e c_e^*\right) = \left(\frac{dc_e}{dt}\right) c_e^* + c.c.$$

$$= -\Gamma\rho_{ee} + i\frac{\Omega}{2}\left(\rho_{ge} - \rho_{eg}\right)$$

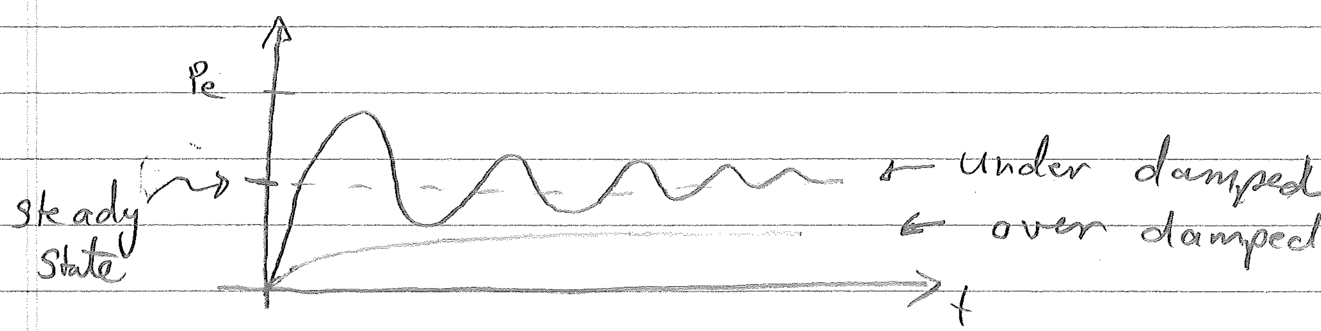
$$\frac{d}{dt}\rho_{eg} = \frac{d}{dt}\left(c_e c_g^*\right) = \frac{dc_e}{dt} c_g^* + c_e\left(\frac{dc_g}{dt}\right)^*$$

$$= \left(i\Delta - \frac{\Gamma}{2}\right)\rho_{eg} + i\frac{\Omega}{2}\left(\rho_{gg} - \rho_{ee}\right)$$

$$\frac{d\rho_{gg}}{dt} = \frac{d}{dt}\left(c_g c_g^*\right) + \left.\frac{d\rho_{gg}}{dt}\right|_{\text{feeding}} = -\frac{d\rho_{ee}}{dt}$$

$$= +\Gamma\rho_{ee} - i\frac{\Omega}{2}\left(\rho_{ge} - \rho_{eg}\right)$$

These equations are known as the "Master equation" for the driven-damped two-level atom. The solution to these equations show damped Rabi Oscillations



Master Equation

The master equation can be expressed in operator form on the density operator. In the absence of dissipations $|\psi\rangle$ evolves according to the Schrodinger Eq: $\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar} H|\psi\rangle$

The statistical mixture: $\hat{\rho} = \sum p_j |\psi(j)\rangle\langle\psi(j)|$

$\Rightarrow$ Hamiltonian evolution of $\hat{\rho}$

$$\frac{d\hat{\rho}}{dt} = \sum p_j \left(\frac{\partial}{\partial t}|\psi\rangle\right)\langle\psi| + \sum p_j |\psi\rangle\left(\frac{\partial}{\partial t}\langle\psi|\right)$$

$$= \sum_j p_j \left(\frac{-i}{\hbar} H|\psi\rangle\langle\psi| + \frac{i}{\hbar}|\psi\rangle\langle\psi| H\right)$$

$$\boxed{\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}\right]}$$ Schrödinger equation for $\hat{\rho}$

Including dissipation (open quantum system)

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \mathcal{L}(\hat{\rho})$$

"Liouvillian"

$$\mathcal{L}(\hat{\rho}) \equiv -\frac{1}{2}\left(\hat{L}^\dagger\hat{L}\hat{\rho} + \hat{\rho}\hat{L}^\dagger\hat{L}\right) + \hat{L}\rho\hat{L}^\dagger$$

"Lindblad form" $\hat{L}$ = Lindblad jump operator

For two-level atom

$$\hat{L} = \sqrt{\Gamma}\,\hat{\sigma}_- = \sqrt{\Gamma}\,|g\rangle\langle e|$$

jump atom from $|g\rangle\langle e|$

$$\hat{L}^\dagger\hat{L} = \Gamma\,\hat{\sigma}_+\hat{\sigma}_- = \Gamma\,|e\rangle\langle e|$$

Transition rate $\gamma_{f\leftarrow i} = |\langle f|\hat{L}|i\rangle|^2$

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] - \Gamma(\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_-) + \Gamma\sigma_-\rho\sigma_+$$

$$= \underbrace{-\frac{i}{\hbar}[\hat{H},\hat{\rho}]}_{\text{Hamiltonian evolution}} \underbrace{-\Gamma(|e\rangle\langle e|\rho + \rho|e\rangle\langle e|)}_{\text{decay}} + \underbrace{\Gamma|g\rangle\langle g|\overset{\langle e|\rho|e\rangle}{}}_{\text{feeding}}$$