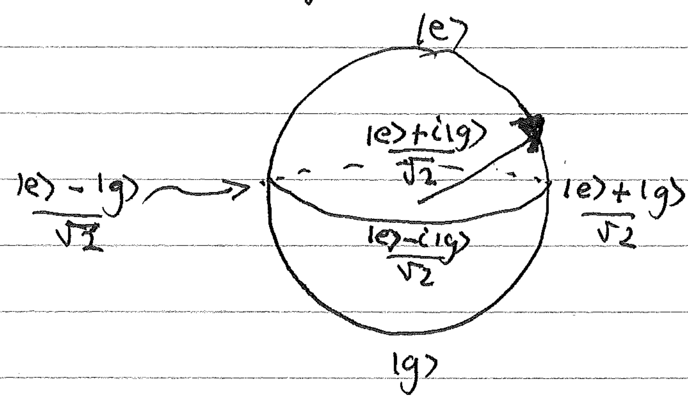
Notes 15: Optical Bloch Equations and Two-level atomic response

Bloch Vector

We have seen that a two-level atom described by a state $|\psi\rangle = c_g|g\rangle + c_e|e\rangle$ is isomorphic to a pseudospin - $1/2$ particle with

$$|e\rangle = |\uparrow\rangle$$
$$|g\rangle = |\downarrow\rangle$$

The different states then correspond points on the surface of the "Bloch Sphere"



The states is then in 1-to-1 correspondence with the unit vector $|\vec{R}| = 1$

$$\vec{R} = \langle \hat{\vec{\sigma}} \rangle = \langle \hat{\sigma}_x \rangle \vec{e}_x + \langle \hat{\sigma}_y \rangle \hat{e}_y + \langle \hat{\sigma}_z \rangle \vec{e}_z$$

direction of pseudospin

Traditionally, this is known as the "Bloch Vector"

$$\vec{R} = u\vec{e}_x + v\vec{e}_y + w\vec{e}_z$$

The components of $\vec{R}$ are related to the components of the density matrix

$$\hat{\sigma}_x = |e\rangle\langle g| + |g\rangle\langle e|$$
$$\hat{\sigma}_y = -i|e\rangle\langle g| + i|g\rangle\langle e|$$
$$\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|$$

$$\Rightarrow \quad u = c_e^* c_g + c_g^* c_e = \rho_{ge} + \rho_{eg} = 2\,\mathrm{Re}(\rho_{ge})$$
$$v = -i c_e^* c_g + i c_g^* c_e = \frac{\rho_{ge} - \rho_{eg}}{i} = 2\,\mathrm{Im}(\rho_{ge})$$
$$w = |c_e|^2 - |c_g|^2 = \rho_{ee} - \rho_{gg}$$

- The u, v components characterize the <u>coherence</u>
- The w component characterizes the <u>population inversion</u>

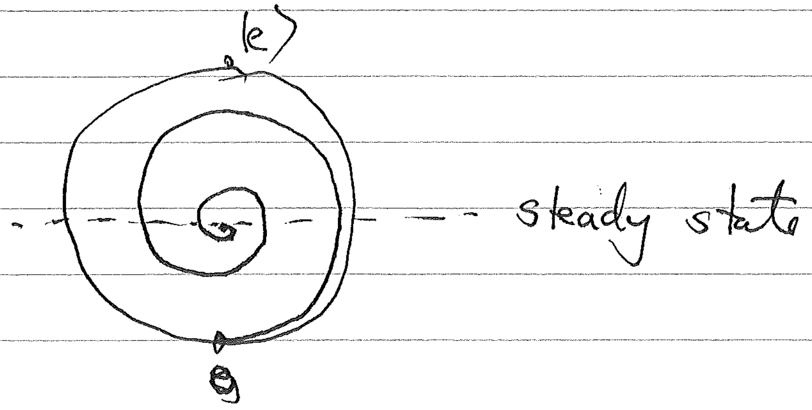
- For a <u>pure state</u>

$$|\vec{R}|^2 = u^2 + v^2 + w^2 = 1$$

- For a <u>mixed state</u> we must ensemble average over different Bloch vectors

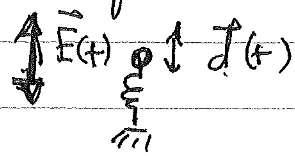
$$\Rightarrow \quad |\vec{R}| < 1$$

Damped Rabi Oscillations



Dipole Response

Classically, when we drive a bound charge we induce an oscillating dipole. the same is true quantum mechanically



$$\vec{E}(t) = \vec{E}_0 \cos\omega_L t \qquad \vec{d}(t) = ?$$

The expectation value of the dipole operator

$$\langle \hat{d}(t) \rangle = \mathrm{Tr}\left(\left(d_{eg} |e\rangle\langle g| + d_{ge} |g\rangle\langle e| \right) \hat{\rho}(t) \right)$$

$$= d_{eg} \left(\rho_{ge}(t) + \rho_{eg}(t) \right)$$

Taking d_{eg} Real

In the rotating frame $\rho_{eg}(t) = \tilde{\rho}_{eg}(t)\, e^{i\omega_L t}$

$$\Rightarrow \langle \hat{d}(t) \rangle = d_{eg} \cdot 2\,\mathrm{Re}\Big(\underbrace{\tilde{\rho}_{eg}(t)}_{= \frac{u+iv}{2}}\, e^{-i\omega_L t} \Big)$$

(Bloch vector in Rotating frame)

$$= \underbrace{d_{eg}\, u(t) \cos\omega_L t}_{\text{In phase with drive}} + \underbrace{d_{eg}\, v(t) \sin\omega_L t}_{\text{In quadrature with drive}}$$

Lecture 5b: Two-level atomic response

• "Natural Lineshape"

Recall the time evolution of mean dipole-moment ("Lab frame")

$$\langle \vec{d}(t) \rangle = \vec{d}_{eg} \left(u(t) \cos\omega_L t - v(t) \sin\omega_L t \right)$$

$u(t) \Rightarrow$ In phase with $\vec{E} \Rightarrow$ Index of refraction

$-v(t) \Rightarrow$ In quadrature with $\vec{E} \Rightarrow$ Absorption/emission

$$\frac{u+iv}{2} = \rho_{ge}$$

Steady state $\Rightarrow$ After transients decay $\Rightarrow \dot{\rho}_{s.s.} = 0$

$\Rightarrow$ Steady state coherences $\dot{\rho}_{ge} = 0$

$$\Rightarrow 0 = \left(-i\Delta - \frac{\Gamma}{2}\right) \rho_{ge}^{s.s.} + i\frac{\Omega}{2}\left(\rho_{ee}^{s.s.} - \rho_{gg}^{s.s.}\right)$$

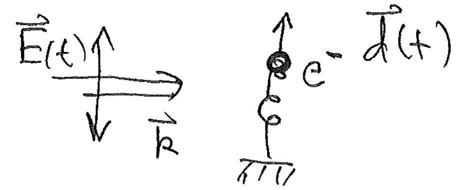
$$\Rightarrow \boxed{\rho_{ge}^{s.s.} = \left(\frac{\Omega/2}{\Delta - i\frac{\Gamma}{2}}\right)\left(\rho_{ee}^{s.s.} - \rho_{gg}^{s.s.}\right)}$$

Suppose in steady state $\rho_{e.e.}^{s.s} \approx 0$ (We come back to see when true)

$\Rightarrow \rho_{gg} \approx 1$

$\Rightarrow$ For weak excitation $\boxed{\rho_{g.e.}^{s.s.} = \left(\frac{-\Omega/2}{\Delta - i\frac{\Gamma}{2}}\right)}$ "Complex Lorentzian"

"Linear response" $\Rightarrow$ Classical S.H.O.



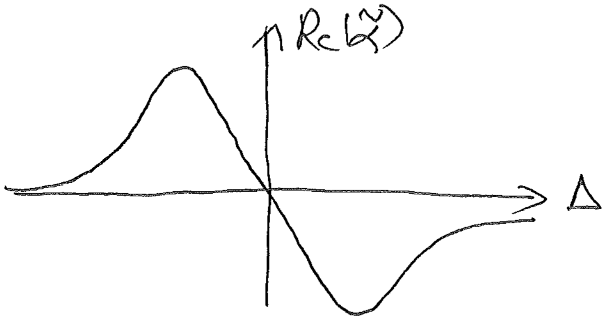
$$\vec{d}(t) = \mathrm{Re}(\underset{\uparrow \text{polarizability}}{\tilde{\alpha}} \vec{E}_0 e^{-i\omega_L t})$$

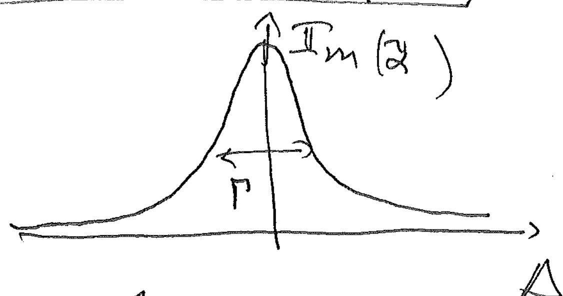
$$\langle \vec{d}(t) \rangle_{s.s.} = \mathrm{Re}\left[\vec{d}_{eg} \underbrace{(u^{s.s.} - i v^{s.s.})} e^{-i\omega_L t} \right]$$

$$2\rho_{eg}^{s.s.} = \frac{-\Omega}{\Delta + i\Gamma/2} = \frac{-\vec{d}_{eg} \cdot \vec{E}}{\hbar(\Delta + i\frac{\Gamma}{2})}$$

$\therefore$ Quantum polarizability

$$\boxed{\tilde{\alpha} = \frac{-|\vec{d}_{eg}|^2}{\hbar(\Delta + i\frac{\Gamma}{2})} = \frac{+|\vec{d}_{eg}|^2}{\hbar} \left(\frac{-\Delta + i\frac{\Gamma}{2}}{\Delta^2 + \frac{\Gamma^2}{4}} \right)}$$





Lorentzian

Natural Line width $= \Gamma$

$\Rightarrow$ Absorption strong when $\Delta < \Gamma$

Rate equations

In the regime when the transient dynamics have damped out and the coherences reach steady state, we can look at population dynamics.
Plugging in the coherences ~~from~~ is s.s.

$$\Rightarrow \dot{\rho}_{ee} = -\Gamma \rho_{ee} - \Omega \, \mathrm{Im}(\rho_{ge}^{s.s.})$$

$$\dot{\rho}_{ee} = -\Gamma \rho_{ee} + \left(\frac{(\Omega^2/4)\Gamma}{\Delta^2 + \frac{\Gamma^2}{4}} \right) (\rho_{ee} - \rho_{gg})$$

Interpretation : Population rate equations

$$N_e \equiv \rho_{ee} \qquad N_g \equiv \rho_{gg}$$

$$\left(\frac{\Omega^2/4}{\Delta^2 + \frac{\Gamma^2}{4}} \right) \Gamma \equiv \Gamma_{stim} = \text{stimulated absorption}$$
$$= \text{stimulated emission}$$

Check: Fermi's Golden rule:

$$\Gamma_{stim} = \frac{2\pi}{\hbar^2} |\langle e| \hat{H}_{int}^{(+)} |g\rangle|^2 \mathcal{D}(\omega_L)$$

density of states

$$\hat{H}_{int}^{(+)} = -\frac{\hbar \Omega}{2} \hat{\sigma}_+ e^{-i\omega_L t}$$

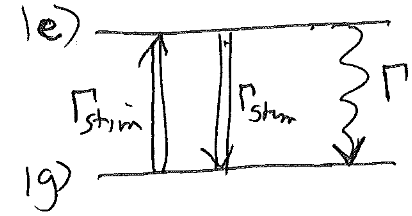
$$\mathcal{D}(\omega_L) = \text{atomic lineshape (normalized)} = \frac{\Gamma/2\pi}{\Delta^2 + \frac{\Gamma^2}{4}}$$

$$\Rightarrow \Gamma_{stim} = \frac{2\pi}{\hbar^2} \left(\frac{\hbar^2 \Omega^2}{4} \right) \frac{\Gamma/2\pi}{\Delta^2 + \frac{\Gamma^2}{4}} = \frac{\frac{\Omega^2 \Gamma}{4}}{\Delta^2 + \frac{\Gamma^2}{4}} \checkmark$$

$\Rightarrow$ Rate equations

$$\dot{N}_e = -(\Gamma + \Gamma_{stim}) N_e + \Gamma_{stim} N_g$$

$$\dot{N}_g = -\Gamma_{stim} N_g + (\Gamma + \Gamma_{stim}) N_e$$



$$\dot{N}_e = -\dot{N}_g$$
$$N_e + N_g = 1$$

Steady state $\Rightarrow$ detailed balance $\dot{N}_e = \dot{N}_g = 0$

$\Rightarrow$ Steady state populations

$$\dot{N}_e = 0 = -(\Gamma + \Gamma_{stim}) N_e^{s.s.} + \Gamma_{stim}(1 - N_e^{s.s.})$$

$$\Rightarrow \boxed{N_e^{s.s.} = \frac{\Gamma_{stim}}{\Gamma + 2\Gamma_{stim}} = \frac{S/2}{1+S}} \qquad \boxed{N_g^{s.s.} = \frac{1+S/2}{1+S}}$$

Here I have defined:

<u>Saturation parameter</u>: $\boxed{S \equiv \frac{2\Gamma_{stim}}{\Gamma} = \frac{\Omega^2/2}{\Delta^2 + \frac{\Gamma^2}{4}}}$

For a given detuning and "oscillator strength", the saturation parameter determines the intensity ($\sim \Omega^2$) at which we pump substantial population into the excited state. When $S \ll 1$

$$N_g \simeq 1 - S/2 \quad , \quad N_e \simeq S/2 \quad \Rightarrow \text{Linear regime}$$

when $s \to \infty$, $N_e^{s.s.} \approx N_g \approx \frac{1}{2}$

$\Rightarrow$ the transition is "saturated".

<u>Resonant behavior:</u> $\Delta = 0$

$$S_0 = \frac{2\Omega^2}{\Gamma^2} \Rightarrow \underline{S = 1 \text{ when } \Omega = \frac{\Gamma}{\sqrt{2}}}$$

Since $\Omega = \frac{d_{eg}E}{\hbar} \propto E \Rightarrow \Omega^2 \propto E^2 \propto I$ (<u>intensity</u>)

$$\Rightarrow \boxed{S_0 = \frac{I}{I_{sat}} \text{ where } I_{sat} \equiv \text{saturation intensity}}$$

$$I_{sat} = \frac{I}{S_0} = \frac{I}{\frac{2\Omega^2}{\Gamma^2}} = \frac{\hbar^2\Gamma^2}{2d_{eg}^2}\frac{I}{E^2} = \left(\frac{\hbar^2\Gamma^2}{2d_{eg}^2}\right)\left(\frac{\frac{c}{8\pi}E^2}{E^2}\right)$$

$$I_{sat} = \frac{c}{16\pi}\frac{\hbar^2\Gamma^2}{d_{eg}^2}$$

c.g.s units

[Aside: We will see that $\Gamma = \frac{4}{3}\left(\frac{\omega_{eg}}{c}\right)^3 \frac{d_{eg}^2}{\hbar}$

$$\Rightarrow d_{eg}^2 = \frac{3\hbar\Gamma}{4}\left(\frac{c}{\omega_{eg}}\right)^3]$$

$$\Rightarrow I_{sat} = (\hbar\omega_{eg})\left(\frac{1}{6\pi\left(\frac{c}{\omega_{eg}}\right)^2}\right)\left(\frac{\Gamma}{2}\right) = \frac{\text{Energy}}{\text{time}\cdot\text{Area}}$$

Here I have used $\sigma_{abs} = 6\pi\left(\frac{c}{\omega_{eg}}\right)^2$ = absorption cross-section for 2-level atom

$$\boxed{I_{sat} = \frac{\hbar\omega_{eg}\frac{\Gamma}{2}}{\sigma_{abs}}}$$

Off-resonance

$$\boxed{s = \frac{s_0}{1+\frac{4\Delta^2}{\Gamma^2}} = \frac{I/I_{sat}}{1+\frac{4\Delta^2}{\Gamma^2}}}$$

saturation falls off like $\frac{\Gamma^2}{\Delta^2}$ for $\Delta \gg \Gamma$

Absorption cross-section and scattering

Recall definition of cross-section:
Given incident intensity I, the absorbed power is

$$P_{abs} = I\,\sigma_{abs}$$

$$= (\text{Rate of absorption}) \times \hbar\omega$$

Rate of absorption $= \Gamma_{stim} = \frac{s}{2}\Gamma$

$$\Rightarrow \frac{s\Gamma}{2}\hbar\omega = I\,\sigma_{abs} = \left(\frac{I/I_{sat}}{1+\frac{4\Delta^2}{\Gamma^2}}\right)\frac{\Gamma}{2}\hbar\omega$$

$\Rightarrow$ On resonance $\quad I\,\sigma_{abs} = \left(\frac{I}{I_{sat}}\right)\frac{\Gamma}{2}\hbar\omega_{eg}$

$$\Rightarrow \boxed{I_{sat} = \frac{\hbar\omega_{eg}}{\sigma_{abs}}\frac{\Gamma}{2}}$$

As before

For two level atom rate of absorption $\rightarrow$ Photon is scattered

$$\Rightarrow \text{Scattering rate} \quad \boxed{\frac{s\Gamma}{2} = \frac{I\,\sigma_{abs}}{\hbar\omega}}$$

Saturation and Power Broadening

Let us return to the steady-state coherence and polarizability. The lineshape we found was for weak excitation, i.e., $s \ll 1$

More generally

$$\rho_{ge}^{s.s.} = \left(\frac{+\Omega/2}{\Delta + i\frac{\Gamma}{2}} \right) \left(\rho_{ee}^{s.s.} - \rho_{gg}^{s.s.} \right)$$

$\Rightarrow$ Steady state Bloch vector components

$$W^{s.s.} = \rho_{e.e}^{s.s.} - \rho_{g.g.}^{s.s.} = \boxed{\frac{-1}{1+s}}$$

$$u^{s.s.} = 2\,\mathrm{Re}(\rho_{g.e}^{s.s.}) = \left(\frac{\Omega \Delta}{\Delta^2 + \Gamma^2/4} \right) \left(\frac{1}{1+s} \right)$$

$$\Rightarrow \boxed{u^{s.s.} = -\frac{2\Delta}{\Omega} \frac{s}{1+s}}$$

$$v^{s.s.} = 2\,\mathrm{Im}(\rho_{ge}^{s.s.}) = \left(\frac{\Omega \,\Gamma/2}{\Delta^2 + \Gamma^2/4} \right) \left(\frac{1}{1+s} \right)$$

$$\Rightarrow \boxed{v^{s.s.} = -\frac{\Gamma}{\Omega} \frac{s}{1+s}}$$

For $s \ll 1$
$$\begin{cases} u^{s.s} \simeq -\dfrac{2\Delta}{\Omega} s = \dfrac{\Delta \Omega}{\Delta^2 + \frac{\Gamma^2}{4}} \\ -v^{s.s.} \simeq +\dfrac{\Gamma}{\Omega} s = \dfrac{+\Gamma \Omega/2}{\Delta^2 + \frac{\Gamma^2}{4}} \end{cases}$$

For $s \gg 1$, $w \to 0$ Saturation

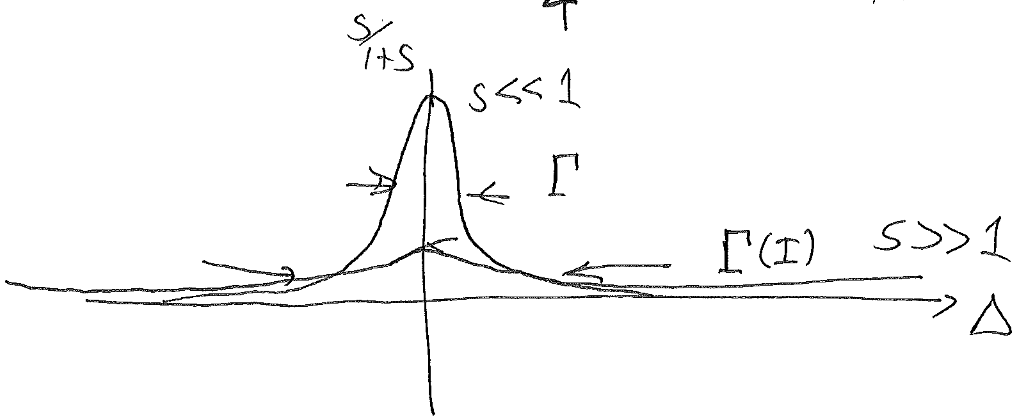
$\Omega \gg \Delta, \Gamma \Rightarrow u, v \to 0$

$\Rightarrow$ Saturation = Completely mixed state

More details:

$$\frac{s}{1+s} = \frac{I/I_{sat}}{1+\frac{4\Delta^2}{\Gamma^2} + I/I_{sat}} = \frac{\frac{2\Omega^2}{\Gamma^2}}{1+\frac{4\Delta^2}{\Gamma^2} + \frac{2\Omega^2}{\Gamma^2}}$$

$$\Rightarrow \frac{s}{1+s} = \frac{\Omega^2/2}{\Delta^2 + \frac{\Gamma^2}{4}\left(1 + I/I_{sat}\right)}$$



"Power Broadened line width"

$$\boxed{\Gamma(I) = \Gamma\sqrt{1 + I/I_{sat}}}$$