Physics 531, P.S.# 4 solutions

Problem 1: The Landé Projection theorem

The L.P.T.: $\langle \alpha'; jm' | \hat{\vec{V}} | \alpha; jm \rangle = \dfrac{\langle \alpha'; j | \hat{\vec{J}} \cdot \hat{\vec{V}} | \alpha; j \rangle}{j(j+1)} \langle \alpha'; jm' | \vec{J} | \alpha, j m \rangle$

(a) The L.P.T. states mathematically that all matrix elements of a vector operator $\hat{\vec{V}}$ restricted to states with eigenvalue j are proportional to matrix element of $\hat{\vec{J}}$

i.e., in this subspace $\quad \hat{\vec{V}} = C(\alpha' \alpha; j) \, \hat{\vec{J}}$

constant depending only on j and other non-geometric g-numbers

Geometrically, if we imagine $\vec{J}$ to be a vector, classically the vector $\vec{V}$ along $\vec{J}$ is

$$(\vec{V} \cdot \vec{e}_J) \vec{e}_J \quad \text{where} \quad \vec{e}_J = \frac{\vec{J}}{|\vec{J}|}$$

$$= \left(\frac{\vec{V} \cdot \vec{J}}{|\vec{J}|^2} \right) \vec{J}$$

Here with $|\vec{J}|^2 = j(j+1)$, the "classical" projection is

$$\vec{V}\Big|_{\vec{J}} = \underbrace{\left[\frac{\vec{V} \cdot \vec{J}}{j(j+1)} \right]}_{\text{"Constant"}} \vec{J}$$

(b) Proof of the L.P.T.

(i) Consider $\langle \alpha'; jm | \hat{\vec{J}} \cdot \hat{\vec{V}} | \alpha j m \rangle = \sum_q (-1)^q \langle \alpha'; jm | \vec{J}_q \hat{V}_{-q} | \alpha; jm \rangle$

Insert complete set $\sum_{m'} |\alpha' j m'\rangle\langle \alpha' j m|$

$$= \sum_{q,m'} (-1)^q \langle \alpha'; jm | \hat{J}_q | \alpha'; jm' \rangle \langle \alpha'; jm' | \hat{V}_{-q} | \alpha; jm \rangle$$

By W.E.T.

$$= \sum_{q,m'} (-1)^q \langle jm | 1q\, jm' \rangle \langle jm' | 1\, {-q}\, jm \rangle \langle \alpha' j \| J \| \alpha' j \rangle \langle \alpha' j \| V \| \alpha j \rangle$$

Aside $\langle jm' | 1\, {-q}\, jm \rangle = (-1)^q \langle 1 q\, jm' | jm \rangle$ (Exchange rules)

$$= \sum_{q,m'} \underbrace{\langle jm | 1q\, jm' \rangle \langle 1 q\, jm' | jm \rangle}_{=1} \langle \alpha' j \| J \| \alpha' j \rangle \langle \alpha' j \| V \| \alpha j \rangle$$

$$= \langle \alpha' j \| J \| \alpha' j \rangle \langle \alpha' j \| V \| \alpha j \rangle$$ <u>independent of m</u> (as it should be for a <u>scalar</u>)

(ii) For $\hat{V} = \hat{J}$, $\alpha = \alpha'$: $\langle \alpha; j | \vec{J}^2 | \alpha j \rangle = | \langle \alpha j \| J \| \alpha j \rangle |^2$

$$\langle \alpha; j | \vec{J}^2 | \alpha j \rangle = j(j+1) \checkmark$$

<u>Independent of α</u>

(iii) Apply W.E.T. to $\hat{J}_q$

$$\langle jm' | \hat{J}_q | jm \rangle = \langle j \| J \| j \rangle \langle jm' | 1 q\, jm \rangle$$

$$\Rightarrow \quad \langle jm' | 1 q\, jm \rangle = \frac{\langle jm' | \hat{J}_q | jm \rangle}{\langle j \| J \| j \rangle} = \frac{\langle jm | \hat{J}_q | jm \rangle}{\sqrt{j(j+1)}}$$

(iv) Putting it all together

From W.E.T. $\langle \alpha' j m' | \hat{V}_q | \alpha j m \rangle = \langle \alpha' j \| V \| \alpha j \rangle \langle j m' | 1 q j m \rangle$

$$\langle \alpha' j \| V \| \alpha j \rangle = \frac{\langle \alpha' j | \hat{\vec{J}} \cdot \hat{\vec{V}} | \alpha j \rangle}{\langle j \| J \| j \rangle} \qquad \langle j m' | 1 q j m \rangle = \frac{\langle \alpha' j m' | \hat{J}_q | \alpha j m \rangle}{\langle j \| J \| j \rangle}$$

True for any component q

$$\Rightarrow \boxed{\langle \alpha' j m' | \hat{\vec{V}} | \alpha j m \rangle = \frac{\langle \alpha' j | \hat{\vec{J}} \cdot \hat{\vec{V}} | \alpha j \rangle}{j(j+1)} \langle \alpha' j m' | \hat{\vec{J}} | \alpha j m \rangle}$$

(c) Application: the Landé g-factor

The magnetic dipole moment for the electron is

$$\hat{\vec{\mu}} = -\mu_B (g_\ell \hat{\vec{L}} + g_S \hat{\vec{S}}) \qquad g_\ell = 1 \qquad g_S = 2$$

When restricted to a manifold with definite $\vec{J} = \vec{L} + \vec{S}$ (i.e. the Zeeman interaction is small compared to fine structure)

$$\hat{\vec{\mu}} = \frac{\langle \alpha J | \hat{\vec{\mu}} \cdot \hat{\vec{J}} | \alpha J \rangle}{J(J+1)} \hat{\vec{J}} = -\mu_B \, g_J \, \hat{\vec{J}}$$

where $$g_J = -\frac{\langle \alpha J | \hat{\vec{\mu}} \cdot \hat{\vec{J}} | \alpha J \rangle}{\mu_B J(J+1)} = \frac{\langle \alpha J | (\hat{\vec{L}} \cdot \hat{\vec{J}} + 2 \hat{\vec{S}} \cdot \hat{\vec{J}}) | \alpha J \rangle}{J(J+1)}$$

Aside: $\hat{\vec{L}} \cdot \hat{\vec{J}} = \hat{\vec{L}} \cdot (\hat{\vec{L}} + \hat{\vec{S}}) = \hat{\vec{L}}^2 + \hat{\vec{L}} \cdot \hat{\vec{S}}$

$\hat{\vec{S}} \cdot \hat{\vec{J}} = \hat{\vec{S}} \cdot (\hat{\vec{L}} + \hat{\vec{S}}) = \hat{\vec{S}}^2 + \hat{\vec{L}} \cdot \hat{\vec{S}}$

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Aside continued: $\hat{\vec{J}}^2 = |\hat{\vec{L}} + \hat{\vec{S}}|^2 = \hat{\vec{L}} + \hat{\vec{S}}^2 + 2\hat{\vec{L}}\cdot\hat{\vec{S}}$

$$\Rightarrow \hat{\vec{L}}\cdot\hat{\vec{S}} = \frac{1}{2}(\hat{J}^2 - \hat{L}^2 - \hat{S}^2)$$

$$\Rightarrow \hat{\vec{L}}\cdot\hat{\vec{J}} = \hat{L}^2 + \frac{1}{2}(\hat{J}^2 - \hat{L}^2 - \hat{S}^2)$$

$$\hat{\vec{S}}\cdot\hat{\vec{J}} = \hat{S}^2 + \frac{1}{2}(\hat{J}^2 - \hat{L}^2 - \hat{S}^2)$$

$$\Rightarrow g_J = \frac{L(L+1) + 2S(S+1) + \frac{3}{2}\left(J(J+1) - L(L+1) - S(S+1)\right)}{J(J+1)}$$

where I have replaced the operators $\hat{J}^2, \hat{L}^2, \hat{S}^2$ by their eigenvalues associated with the good quantum numbers of that level.

For the electron $S = 1/2 \Rightarrow S(S+1) = \frac{3}{4}$

Simplify $\Rightarrow$ $$\boxed{g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)} = \frac{3}{2} + \frac{3}{8J(J+1)} - \frac{L(L+1)}{2J(J+1)}}$$

Note when $L=0$ $\quad g_J = 2$ as expected

(d) For the $2P_{1/2}$ state $L=1 \quad J = 1/2 \Rightarrow g_J(2P_{1/2}) = 2/3$

$2P_{3/2}$ state $L=1 \quad J=3/2 \Rightarrow g_J(2P_{3/2}) = 4/3$

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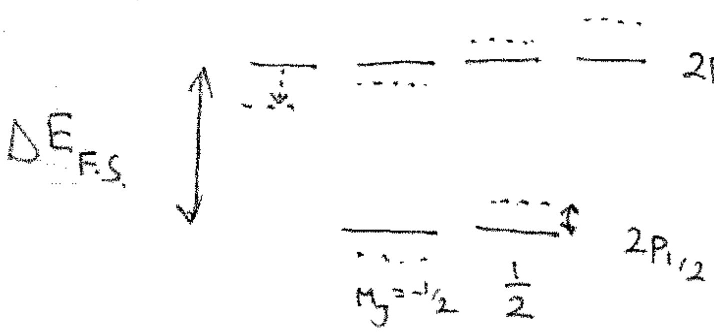
The Zeeman interaction acts only in the manifold nL_J when the shifts are small compared to the fine-structure splitting.



Zeeman shift:
$m_J \, g_J (\mu_B B) = \Delta E_Z$

Require $$\frac{3}{2} g_J(2P_{3/2}) \mu_B B + \frac{1}{2} g_J(2P_{1/2}) \mu_B B \ll \Delta E_{F.S.}$$

$$\Rightarrow \mu_B B \ll \frac{3}{7} \Delta E_{F.S.}$$

The fine-structure splitting between $2P_{3/2}$ and $2P_{1/2}$ is
$\Delta E_{F.S} = (11 \text{ GHz}) h$, and $\mu_B = (1.4 \text{ MHz/Gauss}) h$

$$\Rightarrow \boxed{\text{Require } B \ll \left(\frac{3}{7}\right)\left(\frac{11 \times 10^3}{1.4}\right) G = 3.4 \text{ MGauss} = 3.4 \text{ Tesla}}$$

A huge field!

(e) We want to diagonalize $\hat{H}_{FS} + \hat{H}_{Zeeman}$

In the basis $\{|2p_{1/2}, m_j\rangle, \ |2p_{3/2}, m_j\rangle\}$

As in P.S.#3, the Zeeman Hamiltonian only mixes states with the same m_j.

First let us recall the decomposition of the finestructure in the coupled basis

$$|2p_{1/2}, m_j = \tfrac{1}{2}\rangle = \overset{\leftarrow \text{radial}}{|2p\rangle}\left(\sqrt{\tfrac{1}{3}}\,|m_\ell = 0\rangle \otimes |\uparrow\rangle - \sqrt{\tfrac{2}{3}}\,|m_\ell = 1\rangle|\downarrow\rangle\right)$$

$$|2p_{1/2}, m_j = -\tfrac{1}{2}\rangle = |2p\rangle\left(\sqrt{\tfrac{1}{3}}\,|m_\ell = 0\rangle \otimes |\downarrow\rangle - \sqrt{\tfrac{2}{3}}\,|m_\ell = 1\rangle|\uparrow\rangle\right)$$

$$|2p_{3/2}, m_j = \tfrac{3}{2}\rangle = |2p\rangle|m_\ell = 1\rangle|\uparrow\rangle$$

$$|2p_{3/2}, m_j = \tfrac{1}{2}\rangle = |2p\rangle\left(\sqrt{\tfrac{2}{3}}\,|m_\ell = 0\rangle|\uparrow\rangle + \sqrt{\tfrac{1}{3}}\,|m_\ell = 1\rangle|\downarrow\rangle\right)$$

$$|2p_{3/2}, m_j = -\tfrac{1}{2}\rangle = |2p\rangle\left(\sqrt{\tfrac{2}{3}}\,|m_\ell = 0\rangle|\downarrow\rangle + \sqrt{\tfrac{1}{3}}\,|m_\ell = 1\rangle|\uparrow\rangle\right)$$

$$|2p_{3/2}, m_j = -\tfrac{3}{2}\rangle = |2p\rangle|m_\ell = -1\rangle|\downarrow\rangle$$

Now $\hat{H}_{zeeman} = \mu_B B (\hat{L}_z + 2\hat{S}_z)$

$$\Rightarrow \hat{H}_{zeeman} |2p_{1/2}, m_j = \pm \tfrac{1}{2}\rangle = \pm \mu_B B \sqrt{\tfrac{1}{3}}\ |2p\rangle |m_\ell = 0, m_s = \pm \tfrac{1}{2}\rangle$$

$$\hat{H}_{zeeman} |2p_{3/2}, m_j = \pm \tfrac{3}{2}\rangle = \pm 2 \mu_B B\, |2p_{3/2}, m_j = \tfrac{3}{2}\rangle$$

$$\hat{H}_{zeeman} |2p_{3/2}, m_j = -\tfrac{1}{2}\rangle = \pm \mu_B B \sqrt{\tfrac{2}{3}}\, |2p\rangle |m_\ell = 0, m_s = \pm \tfrac{1}{2}\rangle$$

The matrix of $\hat{H}_{zeeman}$

$$\mu_B B \begin{array}{c} \begin{bmatrix} +\frac{1}{3} & \frac{\sqrt{2}}{3} & & & & \\ \frac{\sqrt{2}}{3} & +\frac{2}{3} & & & & \\ & & -\frac{1}{3} & \frac{\sqrt{2}}{3} & & \\ & & \frac{\sqrt{2}}{3} & -\frac{2}{3} & & \\ & & & & 2 & \\ & & & & & -2 \end{bmatrix} \end{array} \begin{array}{l} |2p_{1/2}\, m_j = \tfrac{1}{2}\rangle \\ |2p_{3/2}, m_j = \tfrac{1}{2}\rangle \\ |2p_{1/2}, m_j = -\tfrac{1}{2}\rangle \\ |2p_{3/2}, m_j = -\tfrac{1}{2}\rangle \\ |2p_{3/2}, m_j = \tfrac{3}{2}\rangle \\ |2p_{3/2}, m_j = -\tfrac{3}{2}\rangle \end{array}$$

Whereas F.S.

$$\hat{H} = \begin{bmatrix} E_{p_{1/2}} & & & & & \\ & E_{p_{3/2}} & & & & \\ & & E_{p_{1/2}} & & & \\ & & & E_{p_{3/2}} & & \\ & & & & E_{p_{3/2}} & \end{bmatrix} : \text{Diagonal}$$

We must diagonal two 2x2 matrices

$$\begin{bmatrix} E_{P_{1/2}} + \frac{\mu_B B}{3} & \frac{\sqrt{2}}{3}\mu_B B \\ \frac{\sqrt{2}}{3}\mu_B B & E_{P_{3/2}} + \frac{2\mu_B B}{3} \end{bmatrix} \begin{array}{l} |2P_{1/2}\rangle|m_J=1/2\rangle \\ |2P_{3/2}\rangle|m_J=1/2\rangle \end{array}$$

and

$$\begin{bmatrix} E_{P_{1/2}} - \frac{\mu_B B}{3} & \frac{\sqrt{2}}{3}\mu_B B \\ \frac{\sqrt{2}}{3}\mu_B B & E_{P_{3/2}} - \frac{2\mu_B B}{3} \end{bmatrix} \begin{array}{l} |2P_{1/2}\rangle|m_J=-1/2\rangle \\ |2P_{3/2}\rangle|m_J=-1/2\rangle \end{array}$$

If we set the zero of energy halfway between the fine structure

$E_{P_{1/2}} = -\Delta E \qquad E_{P_{3/2}} = +\Delta E$

$\Rightarrow$ For $|m_J = 1/2\rangle$

$$H_{int} \doteq \frac{\mu_B B}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \Delta E + \frac{\mu_B B}{6} & 0 \\ 0 & -(\Delta E + \frac{\mu_B B}{6}) \end{bmatrix} + \frac{\sqrt{2}}{3}\mu_B B \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \text{Eigenvalues } \frac{\mu_B B}{2} \pm \sqrt{\left(\Delta E + \frac{\mu_B B}{6}\right)^2 + \frac{2}{9}\mu_B^2 B^2}$$

For $\Delta E \gg \mu_B B \rightarrow$ Eigenvalue $\Delta E + \frac{2}{3}\mu_B B$

$-\Delta E + \frac{1}{3}\mu_B B$

For $|m_J = -1/2\rangle$

$$H_{int} = -\frac{\mu_B B}{2} + \begin{bmatrix} -\Delta E + \mu_B B/6 & 0 \\ 0 & \Delta E - \mu_B B/6 \end{bmatrix} + \frac{\sqrt{2}}{3}\mu_B B \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\Rightarrow$ Eigenvalues $-\frac{\mu_B B}{2} \pm \sqrt{\left(\Delta E - \frac{\mu_B B}{6}\right)^2 + \frac{2}{9}(\mu_B B)^2}$

$\Delta E \gg \mu_B B \Rightarrow$ Eigenvalues $\Delta E - \frac{2}{3}\mu_B B$

$-\Delta E - \frac{1}{3}\mu_B B$

Compare this to Landé projection

Zeeman shift for level $|J, m_J\rangle$

$= g_J m_J \mu_B B \qquad g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}$

$P_{1/2} \Rightarrow g_J = \frac{2}{3} \qquad P_{3/2} \Rightarrow g_J = \frac{4}{3}$

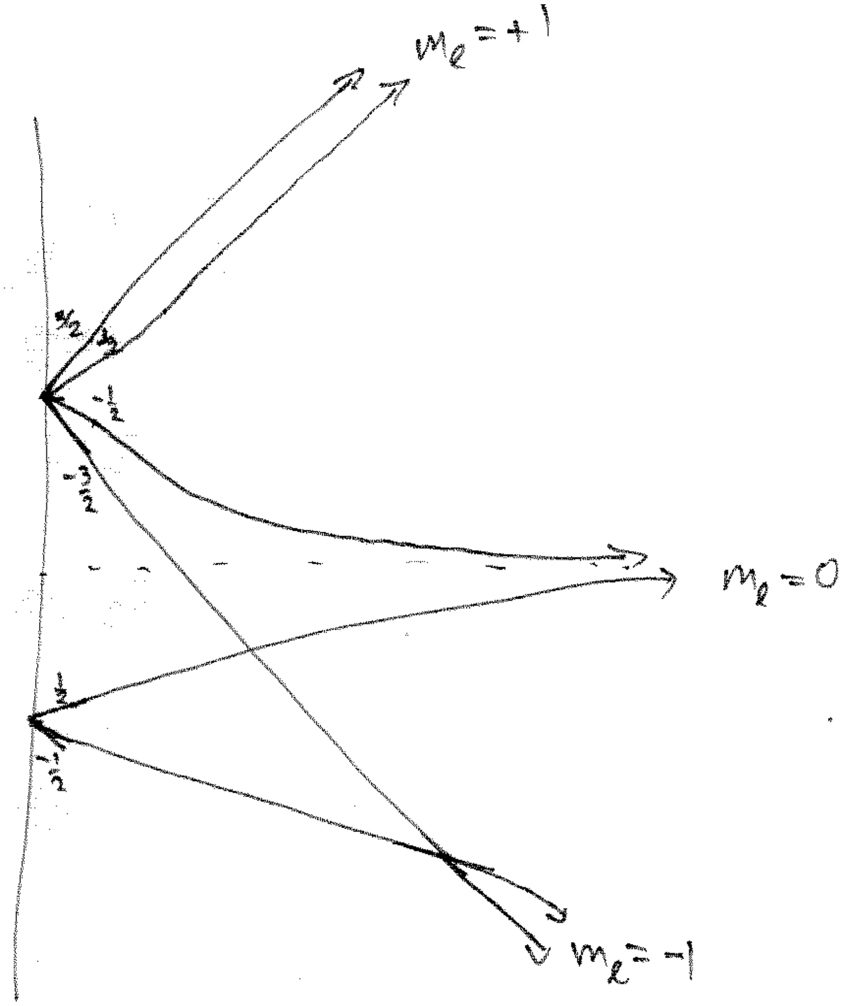
$\Rightarrow$ Shift of $|P_{1/2}, m_J = \pm\frac{1}{2}\rangle = \pm\frac{1}{3}\mu_B B$

Shift of $|P_{3/2}, m_J = \pm 1/2\rangle = \pm\frac{2}{3}\mu_B B$

Agrees !

Also: Shift $|P_{3/2}, m_J = \pm\frac{3}{2}\rangle = \frac{4}{3}\left(\pm\frac{3}{2}\right)\mu_B B = \pm 2\mu_B B$ ✓

Sketch of energy level diagram



- For "small" B, coupled representation, linear shift with m_j

- For very "large" B, uncoupled representation, $m_\ell = 1, 0, -1$ plus spin $|\uparrow\rangle$ or $|\downarrow\rangle$

Problem 2

Spontaneous decay rate on a dipole transition $|\psi_e\rangle \to |\psi_g\rangle$ summed over all possible final states

$$\Gamma = \frac{4}{3\hbar} k^3 \sum_g |\langle \psi_g | \hat{d} | \psi_e \rangle|^2$$

Hydrogen, including fine-structure $(n L J M_J) \to \sum_{M_J'} (n' L' J' M_J')$

$$\Gamma = \frac{4}{3\hbar} k^3 \sum_{M_J'} |\langle n' L' J' M_J' | \hat{d} | n L J M_J \rangle|^2$$

(a) Expand in spherical basis $\hat{d} = \sum_q \vec{e}_q^{\,*} \hat{d}_q$ and use W.E.T

$$\Gamma = \frac{4}{3\hbar} k^3 |\langle n' L' J' \| d \| n L J \rangle|^2 \underbrace{\sum_{q M_J'} |\langle J' M_J' | 1 q \, J M_J \rangle|^2}_{= 1 \text{ by normalization}}$$

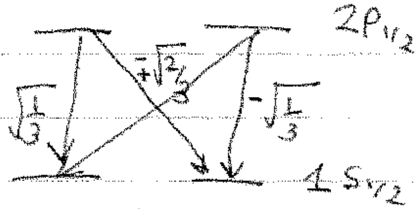
$$\Rightarrow \boxed{\Gamma = \frac{4}{3\hbar} k^3 |\langle n' L' J' \| d \| n L J \rangle|^2 \quad \text{independ of } M_J}$$

This make sense physically. The vacuum is isotropic and will not care what direction the angular momentum of the atom is pointing in.

Aside: Note that $\Gamma \propto \omega^3$. Thus high frequency transitions decay much more rapidly than low frequency transitions. This make sense from classical electromagnetic theory. The Larmor power (rate of energy rated from an oscillating dipole) goes as ω^3.

(b) Life of of $2p_{1/2}$ in Hydrogen

Allowed transitions



Note branching ratios for decay:

$\frac{1}{3} + \frac{2}{3} = 1$ ✓

(Note Lamb shift puts $2S_{1/2}$ above $2P_{1/2}$)

From part (a), we need only consider one initial M_J since Γ is the same for all

$$\Rightarrow \Gamma(2p_{1/2}) = \frac{4k^3}{3\hbar} |\langle 1S_{1/2} \| d \| 2P_{1/2} \rangle|^2$$

To calculate the reduced matrix element, pick some allowed transition and use the W.E.T.

$$\underbrace{\langle 1s_{1/2}, M_J = \tfrac{1}{2} | \hat{d}_z | 2P_{1/2}, M_J = \tfrac{1}{2} \rangle}_{} = \langle 1s_{1/2} \| d \| 2p_{1/2} \rangle \underbrace{\langle \tfrac{1}{2} \tfrac{1}{2} | 1 0 \tfrac{1}{2} \tfrac{1}{2} \rangle}_{= \frac{-1}{\sqrt{3}}}$$

Must uncouple spin + orbit ang. mom.

$$|2P_{1/2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}} |2p, 0\rangle |\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}} |2p, 1\rangle |-\tfrac{1}{2}\rangle \quad \text{(From P.S. \# 1 prob 3)}$$

$$|1S_{1/2}, \tfrac{1}{2}\rangle = |1s, 0\rangle |\tfrac{1}{2}\rangle$$

$$\Rightarrow \langle 1s_{1/2}, \tfrac{1}{2} | \hat{d}_z | 2p_{1/2}, \tfrac{1}{2} \rangle = \frac{1}{\sqrt{3}} \langle 1s, 0 | \hat{d}_z | 2p, 0 \rangle = \frac{-e}{\sqrt{3}} \langle 1s, 0 | \hat{z} | 2p, 0 \rangle$$

Aside: $\langle 1s, 0 | \hat{z} | 2p, 0 \rangle = \int d^3x \, \psi^*_{1s,0}(\vec{x}) \, z \, \psi_{2p,0}(\vec{x})$

$$z = \sqrt{\frac{4\pi}{3}} \, r \, Y^1_0(\theta, \phi)$$

$$\Rightarrow \langle 1s, 0 | \hat{z} | 2p, 0 \rangle = \sqrt{\frac{4\pi}{3}} \int_0^\infty dr \, u_{10}(r) \, r \, u_{20}(r) \int d\Omega \, Y^0_0 Y^1_0 Y^1_0$$

Aside continued:

$$\int_0^\infty dr\, r\, u_{10}(r)\, u_{21}(r) = \frac{a_0}{\sqrt{6}} \int_0^\infty \bar{r}^4 e^{-\frac{3}{2}\bar{r}} d\bar{r} = 1.29\, a_0$$

$$\int d\Omega\, Y_0^0 Y_0^1 Y_0^1 = \frac{1}{\sqrt{4\pi}} \underbrace{\int d\Omega\, Y_0^1 Y_0^1}_{=1} = \frac{1}{\sqrt{4\pi}}$$

$$\Rightarrow \langle 1s_{1/2}\, \tfrac{1}{2} | \hat{d}_z | 2p_{1/2}\, \tfrac{1}{2} \rangle = \left(-\frac{e}{\sqrt{3}}\right)\left(\sqrt{\frac{4\pi}{3}}\right)(1.29\, a_0)\left(\frac{1}{\sqrt{4\pi}}\right)$$

$$= -0.43\, ea_0$$

$$\Rightarrow \langle 1s_{1/2} \| d \| 2p_{1/2} \rangle = \sqrt{3}\, 0.43\, ea_0 = 0.74\, ea_0$$

Thus, $\Gamma(2p_{1/2}) = (0.74)^2 \frac{4}{3\hbar} k^3 (ea_0)^2$

Aside: In atomic units: $\dfrac{\hbar\Gamma}{e^2/a_0} = \frac{4}{3}(ka_0)^3 \left(\frac{\langle d \rangle}{ea_0}\right)^2$

$$k = \frac{\omega}{c} = \frac{\hbar\omega}{\hbar c} = \varepsilon \frac{E_0}{\hbar c} = \varepsilon \frac{e^2}{\hbar c\, a_0} = \alpha \varepsilon \frac{1}{a_0}$$

$$\Rightarrow \hbar\Gamma/(e^2/a_0) = \varepsilon^3 \frac{4}{3} \alpha^3 \left(\frac{\langle d \rangle}{ea_0}\right)^2$$

Here $\dfrac{\langle d \rangle}{ea_0} = 0.74$ $\qquad \varepsilon = \dfrac{\hbar\omega}{E_0} = \dfrac{\frac{3}{8}E_0}{E_0} = \dfrac{3}{8}$

$\alpha = \dfrac{1}{137}$ $\qquad$ (Photon for $n=2 \rightarrow n=1$)

$$\Rightarrow \hbar\Gamma/(e^2/a_0) = 1.5 \times 10^{-8} \qquad \frac{e^2}{a_0} = 27.2\,\text{eV} = 4.1 \times 10^{16}\, s^{-1}$$

$$\Rightarrow \Gamma = 6.2 \times 10^8\, s^{-1}$$

$$\Rightarrow \text{Lifetime } \boxed{\tau = 1/\Gamma = 1.6\ \text{ns}}$$

<u>Problem 3:</u> 3 spherical harmonics

Often, we must calculate the integral of three Spher. Harmonics

$$\langle l_3 m_3 | \hat{Y}^{(l_2)}_{m_2} | l_1 m_1 \rangle = \int d\Omega \, Y^*_{l_3 m} \, Y_{l_2 m_2} \, Y_{l_1 m_1}$$

(a)

$$\underset{\substack{\uparrow \\ \text{by W.E.T.}}}{=} \langle l_3 \| Y^{(l_2)} \| l_1 \rangle \langle l_3 m_3 | l_2 m_2 \, l_1 m_1 \rangle$$

Thus, this integral vanishes unless

$$m_3 = m_2 + m_1$$

$$|l_2 - l_1| \leq l_3 < l_2 + l_1$$

(b) Consider $\langle l_3 0 | \hat{Y}^{(l_2)}_0 | l_1 0 \rangle = \int d\Omega \, Y^*_{l_3 0} \, Y_{l_2 0} \, Y_{l_1 0}$

Now $Y_{l,0} = \sqrt{\frac{2l+1}{4\pi}} \, P_l(\cos\theta)$ ← Legendre polynomial

$$\Rightarrow \langle l_3 0 | \hat{Y}^{(l_2)}_0 | l_1 0 \rangle = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{(4\pi)^3}} \int d\Omega \, P_{l_3}(\cos\theta) \, P_{l_2}(\cos\theta) \, P_{l_1}(\cos\theta)$$

Aside: $P_{\ell_1}(\cos\theta)\, P_{\ell_2}(\cos\theta) = \sum_{\ell'} \langle \ell' 0 | \ell_1 0\, \ell_2 0\rangle^2 P_{\ell'}(\cos\theta)$

$$\therefore \int d\Omega\, P_{\ell_3}(\cos\theta)\, P_{\ell_2}(\cos\theta)\, P_{\ell_1}(\cos\theta) =$$

$$= \sum_{\ell'} \langle \ell' 0 | \ell_1 0\, \ell_2 0\rangle^2 \underbrace{\int P_{\ell_3}(\cos\theta)\, P_{\ell'}(\cos\theta)\, \overset{2\pi\, d(\cos\theta)}{d\Omega}}_{\frac{4\pi}{2\ell_3+1}\delta_{\ell_3 \ell'}}$$

$$\therefore \langle \ell_3 0 | \hat{Y}_0^{(\ell_2)} | \ell_1 0\rangle = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi}}\, \langle \ell_3 0 | \ell_1 0\, \ell_2 0\rangle^2$$

$$= \langle \ell_3 \| \hat{Y}^{(\ell_2)} \| \ell_1 \rangle \langle \ell_3 0 | \ell_1 0\, \ell_2 0\rangle$$

$\Rightarrow$ Can solve for reduced matrix element.

Putting it all together

$$\boxed{\int d\Omega\, Y^*_{\ell_3 m_3} Y_{\ell_2 m_2} Y_{\ell_1 m_1} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi}}\, \langle \ell_3 0 | \ell_2 0\, \ell_1 0\rangle \langle \ell_3 m_3 | \ell_2 m_2\, \ell_1 m_1\rangle}$$