Physics 521: Problem Set #1 Solutions

Problem 1: Hydrogenic atoms in atomic units

Two oppositely charged particles:

- charge 1 (negative) $\quad q_1 = -Z_1 e, \quad$ mass m_1
- Charge 2 (positive) $\quad q_2 = Z_2 e, \quad$ mass m_2

Coulomb interaction: $V(r) = \dfrac{q_1 q_2}{r} = -Z_1 Z_2 \dfrac{e^2}{r}$

Relative motion Hamiltonian: $\hat{H} = \dfrac{P^2}{2\mu} + V(r)$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{(reduced mass)}$$

$\boxed{\text{Characteristic Units}}$: scales determined by $\mu, q_1, q_2, \hbar$

Length: a_c, Momentum: $P_c = \dfrac{\hbar}{a_c}$, Energy: E_c

Relate: $$E_c = \underbrace{\frac{P_0^2}{\mu}}_{\text{kinetic energy}} = \underbrace{\frac{q_1 q_2}{a_c}}_{\text{Coulomb energy}} = \frac{\hbar^2}{a_c^2 \mu}$$

Length: $$\boxed{a_c = \frac{\hbar^2}{\mu q_1 q_2} = \frac{m_e}{\mu Z_1 Z_2}\left(\frac{\hbar^2}{m_e e^2}\right) = \frac{m_e}{\mu Z_1 Z_2}(0.53\,\text{Å})}$$

Momentum $$\boxed{P_c = \frac{\hbar}{a_c} = \left(\frac{\mu Z_1 Z_2}{m_e}\right)\frac{\hbar}{0.53\,\text{Å}} = \frac{\mu Z_1 Z_2}{m_3}\, 2\times 10^{-19}\ \frac{g\,cm}{s}}$$

Problem 3: Finite nuclear size effect

(a) In the standard Hamiltonian for atoms the nucleus is treated as a point charge. A more detailed model treats it as a uniformly charged sphere, radius R, so that the charge density

$$\rho(r) = \begin{cases} \left(\frac{3}{4\pi R^3}\right) & r<R \\ 0 & r>R \end{cases}$$

Then by Gauss's law, the electric field created by this charge density is $\oint E \cdot d\vec{a} = 4\pi Q_{enc}(r)$

$$E(r) = \frac{Q_{enc}(r)}{r^2} = \begin{cases} \frac{r}{R^3} Ze & r<R \\ \frac{Ze}{r^2} & r>R \end{cases}$$

The electrostatic potential, with ground @ ∞

$$\Phi(r) = -\int_\infty^r E(r)\,dr = -\frac{Ze}{r} \qquad r>R$$

$$= -\int_R^r \frac{rZe}{R^3}\,dr + \frac{Ze}{R} \qquad r<R$$

$$= -\frac{1}{2}\frac{r^2 Ze}{R^3} + \frac{3}{2}\frac{Ze}{R} \qquad r<R$$

Potential energy

$$V(r) = -e\Phi(r) = \begin{cases} -\frac{3}{2}\frac{Ze}{R}\left(1-\frac{r^2}{3R^2}\right) & r<R \\ -\frac{Ze^2}{r} & r>R \end{cases}$$

Energy $\boxed{E_c = \frac{q_1 q_2}{a_c} = \left(\frac{\mu Z_1^2 Z_2^2}{m_e}\right)\left(\frac{m_e e^4}{\hbar^2}\right) \doteq \left(\frac{\mu Z_1^2 Z_2^2}{m_e}\right) 27.2\,eV}$

Time: $\boxed{t_c = \frac{\hbar}{E_c} = \left(\frac{m_e}{\mu Z_1^2 Z_2^2}\right)\frac{\hbar}{E_0} = \left(\frac{m_e}{\mu Z_1^2 Z_2^2}\right) 2.4 \times 10^{-17}\,s}$

~10 atto seconds

Internal Electric field @ particle 1 $\boxed{\mathcal{E}_c = \frac{q_2}{a_c^2} = \left(\frac{\mu Z_1 Z_2}{m_e}\right)^2 \frac{Z_2 e}{a_0^2} \simeq \left(\frac{\mu^2 Z_1^2 Z_2^3}{m_e^2}\right) 5 \times 10^9 \frac{V}{cm}}$

Electric dipole moment $d_c = \left(\frac{q_1 - q_2}{2}\right) a_c = \left(\frac{Z_1 + Z_2}{2}\right)\left(\frac{m_e}{\mu}\right)\left(\frac{1}{Z_1 Z_2}\right) e a_0$

Aside $e a_0 = 8.5 \times 10^{-30}$ C-m $= 2.54$ Debye

1 debye $= 10^{-10}$ esu $\cdot$ Å

Speed (units of c) $\boxed{\frac{v_c}{c} = \frac{a_c}{c t_c} = \frac{a_c}{c \frac{\hbar}{E_c}} = \frac{q_1 q_2}{\hbar c} = (Z_1 Z_2)\frac{e^2}{\hbar c} = (Z_1 Z_2)\,\alpha}$ ← fine structure constant

Internal B-field @ particle 1 $B_c = \frac{v_c}{c} E_c = (Z_1 Z_2)(\alpha \mathcal{E}_c) = \left(\frac{\mu^2 Z_1^3 Z_2^4}{m_e^2}\right) \boxed{\frac{e}{a_0^2}\alpha}$

~10^5 Gau[ss]

Magnetic dipole moment $\mu_c = \frac{\text{Current} \times \text{Area}}{c} = \frac{q_1}{E_c}\frac{a_c^2}{c}$

2× μ_B = Bohr magneton

$$\boxed{\Rightarrow \mu_c = \left(\frac{a_c}{t_c c}\right)(q_1 a_c) = \alpha d_c = \left(\frac{Z_1+Z_2}{2}\right)\left(\frac{m_e}{\mu}\right)\left(\frac{e\hbar}{mc}\right)}$$

Now for each case given:

(i) Hydrogen defines "atomic units":

- $a_c = a_0 = 0.53\ \text{Å}$ Bohr radius
- $E_c = E_0 = 27.2\ \text{eV}$ Hartree
- $t_c = 2.4 \times 10^{-17}\ s$
- $\mathcal{E}_c = 5 \times 10^9\ \frac{V}{cm} = 1.7 \times 10^7\ \frac{\text{statV}}{cm}$
- $B_c = \alpha E_c = 1.2 \times 10^5$ Gauss (Not standard atomic unit)
- $d_c = 2.5 \times 10^{-18}$ cgs $= 2.5$ debye
- $\mu_c = 2\mu_B = \alpha d_c = 1.8$ ergs/Gauss $= 1.8 \times 10^{-24}$ Joule/Tesla

(ii) Heavy ion: $Z_1 = 1,\ Z_2 = 50,\ \mu \simeq m_e$

(iii) Muonium: $m_1 \simeq 200\ m_e$ $m_2 = m_p \simeq 2000 m_p$, $\mu \cong 180 m_e$

(iv) Positronium: $Z_1 = Z_2 = 1,\ m_1 = m_2 = m_e,\ \mu = \frac{m_e}{2}$

Summary table: Characteristic Units in a.u.

	a_c	E_c	t_c	p_c	v_c/c	$\mathcal{E}_c$	B_c	d_c	μ_c
Hydrogen	1	1	1	1	1	1	1	1	1
Sn^{+49}	$\frac{1}{50}$	2500	$\frac{1}{2500}$	50	50	2500	6.25×10^6	$\frac{1}{2}$	25
Muonium	$\frac{1}{180}$	180	$\frac{1}{180}$	180	1	32400	32400	$\frac{1}{180}$	$\frac{1}{180}$
Positronium	2	$\frac{1}{2}$	2	$\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{1}{4}$	2	2

From these results we see that

- for heavy elements the electron can be highly relativistic and magnetic effects grow
- Muonium sees huge field because the muon is so close to the nucleus.

Problem 2: Radial Expectation Values for Hydrogen

(a) We seek the mean value of r for $|n, \ell, m\rangle$

$$\langle r \rangle_{n\ell} = \int_0^\infty dr\ r^3 R_{n\ell}(r) \underbrace{\int_0^\infty d\Omega\ |Y_{\ell,m}(\theta,\phi)|}_{=1}$$

where $R_{n\ell}(r) = -\sqrt{N_{n\ell}'} \left(\frac{2Z}{na_0}\right)^{3/2} e^{-\rho/2} \rho^\ell L_{n+\ell}^{2\ell+1}(\rho)$

$$\rho \equiv \frac{2Zr}{na_0} \qquad N_{n\ell} = \text{normalization constant.}$$

By brute force, we use the method given in Appendix 3 of B & J based on the generating function

$$U_p(\rho, s) \equiv (-s)^p \frac{e^{-\rho s/1-s}}{(1-s)^{p+1}} = \sum_{q=p}^{\infty} \frac{L_q^p(\rho)}{q!} s^q$$

$$\therefore \langle r \rangle_{n\ell} = N_{n\ell} \left(\frac{na_0}{2Z}\right) \int_0^\infty d\rho\ \rho^{2\ell+3} e^{-\rho} \left[L_{n+\ell}^{2\ell+1}(\rho)\right]^2$$

Now $\int_0^\infty e^{-\rho} \rho^\alpha U_p(\rho,s) U_p(\rho,t)\, d\rho$

$$= \sum_{q=p}^{\infty} \sum_{q'=p}^{\infty} \frac{s^q t^{q'}}{q!\,q'!} I_{pq,pq'}^{\alpha}$$

where $I_{pq,pq'}^{\alpha} = \int_0^\infty e^{-\rho} \rho^\alpha \left[L_q^p(\rho)\right]^2 d\rho$

Thus, $\langle r \rangle_{n\ell} = N_{n\ell} \frac{na_0}{2Z} I_{pq,pq}^{\alpha}$ $\quad p = 2\ell+1$, $q = n+\ell$, $\alpha = p+2$

Using appendix eq (A3.20)

$$\int_0^\infty e^{-\rho} \rho^\alpha U_p(\rho,s)\, U_p(\rho,t) = \underset{\text{integer}}{\alpha!} \frac{(1-s)^{\alpha+1}}{(1-s)^{p+1}} \frac{(1-t)^{\alpha+1}}{(1-t)^{p+1}} \frac{(st)^p}{(1-st)^{\alpha+1}}$$

Using binomial

$$= \frac{(1-s)^{\alpha+1}}{(1-s)^{p+1}} \frac{(1-t)^{\alpha+1}}{(1-t)^{p+1}} \sum_{k=0}^{\infty} \frac{(\alpha+k)!\,(st)^{k+p}}{k!}$$

Plug in $\alpha = p+2$

$$= (1-s)^2 (1-t)^2 \sum_{k=0}^{\infty} \frac{(p+2+k)!}{k!} (st)^{k+p}$$

$$= (1+s^2-2s)(1+t^2-2t) \sum_{q=p}^{\infty} \frac{(q+2)!}{(q-p)!} (st)^q$$

$$= \sum_{q=p}^{\infty} \sum_{q'=p}^{\infty} \frac{I^{\alpha=p+2}_{pq,pq'}}{q!\,q'!} s^q t^{q'}$$

Equating the coefficients where $q = q'$

$$\Rightarrow I^{\alpha=p+2}_{pq,pq} = (q!)^2 \left[\frac{(q+2)!}{(q-p)!} + \frac{4\,(q+1)!}{(q-p-1)!} + \frac{q!}{(q-2-p)!} \right]$$

$$= \frac{(q!)^3}{(q-p)!} \left[(q+2)(q+1) + 4(q+1)(q-p) + (q-p)(q-p \right.$$

Now, according to the appendix 3 A3.25

with $p = 2l+1$, $q = n+l$ $\Rightarrow$ $N_{nl} = \left. \frac{(q-p)!}{(q!)^3 (2q-p+1)} \right]$

Putting it all together

$$\langle r\rangle_{n\ell} = \frac{na_0}{2Z}\left[\frac{(q+2)(q+1)+4(q+1)(q-p)+(q-p)(q-p-1)}{2q-p+1}\right]$$

Plug in $p=2\ell+1$ $\quad q=n+\ell$

$$\Rightarrow \langle r\rangle_{n\ell} = \frac{na_0}{2Z}\left[3n-\frac{\ell(\ell+1)}{n}\right]$$

$$\boxed{\langle r\rangle_{n\ell} = \frac{n^2a_0}{Z}\left[1+\frac{1}{2}\left(1-\frac{\ell(\ell+1)}{n^2}\right)\right]}$$

phew! What a <u>mess</u>

(b) For circular states $\Rightarrow$ max angular momentum (no radial kinetic energy) $\Rightarrow$ $\ell = n-1$

$$\Rightarrow \langle r\rangle_{n\ell} = n^2\frac{a_0}{Z}\left\{1+\underbrace{\frac{1}{2}\left[1-\frac{n(n-1)}{n^2}\right]}_{\to 0 \text{ as } n\to\infty}\right\}$$

$$\Rightarrow \langle r\rangle_{n\ell} = n^2\frac{a_0}{Z}$$

the Bohr result.

(C) Feynman-Hellman

Consider expectation value as function of parameters

$$\langle n_r, \xi | \hat{H}(\xi) | n_r, \xi \rangle = E_{n_r}(\xi)$$

Take partial w.r.t. some ξ

$$\Rightarrow \left(\frac{\partial}{\partial \xi} \langle n_r \xi|\right) \hat{H}(\xi) |n_r \xi\rangle + \langle n_r, \xi | \hat{H} \left(\frac{\partial}{\partial \xi} |n_r, \xi\rangle\right)$$

$$+ \langle n_r, \xi | \frac{\partial \hat{H}(\xi)}{\partial \xi} | n_r \xi \rangle = \frac{\partial E_{n_r}(\xi)}{\partial \xi}$$

But $\hat{H}(\xi)|n_r \xi\rangle = E_{n_r}(\xi) \, |n_r, \xi\rangle$ and adjoint

$\rightarrow$ First two term $= E_{n_r} \dfrac{\partial}{\partial \xi}\left(\langle n_r \xi | n_r \xi \rangle\right) = 0$

same normalization

$$\Rightarrow \boxed{\langle n_r \xi | \frac{\partial \hat{H}(\xi)}{\partial \xi} | n_r \xi \rangle = \frac{\partial E_{n_r}(\xi)}{\partial \xi}}$$

(d) Now we can use this clever trick

- Let $\xi = e^2 \Rightarrow \quad \dfrac{\partial \hat{H}}{\partial e^2} = \dfrac{-Z}{r} \qquad \dfrac{\partial E_{n_r}}{\partial e^2} = \dfrac{2}{e^2} E$

$$\Rightarrow -Z \langle n_r \xi | \frac{1}{r} | n_r \xi \rangle = \frac{2}{e^2} E_{n_r}(\xi) = \frac{2}{e^2}\left(-\frac{Z^2 e^2}{2 a_0 n^2}\right)$$

$$\rightarrow \boxed{\left\langle \frac{1}{r} \right\rangle_{nl} = \frac{Z}{a_0 n^2}}$$

• Let $\xi = \ell$ $\Rightarrow \frac{\partial \hat{H}}{\partial \ell} = \frac{(2\ell+1)\hbar^2}{2m_e r^2}$ $\frac{\partial E}{\partial \ell} = \frac{-2}{n} E_n$

$$\Rightarrow (\ell+\tfrac{1}{2})\frac{\hbar^2}{m_e}\left\langle \frac{1}{r^2}\right\rangle_{n\ell} = \frac{-2}{n}\frac{Z^2}{-2n^2}\left(\frac{e^2}{a_0}\right)$$

$$\Rightarrow \left\langle \frac{1}{r^2}\right\rangle_{n\ell} = \frac{Z^2}{n^3(\ell+\frac{1}{2})}\frac{m_e e^2}{\hbar a_0} \boxed{= \left(\frac{Z}{a_0}\right)^2 \frac{1}{n^3(\ell+\frac{1}{2})}}$$

• Consider $\left\langle \left[\frac{\partial}{\partial r}, \hat{H}\right]\right\rangle_{n\ell} = \langle n\ell m| \frac{\partial}{\partial r}\hat{H} - \hat{H}\frac{\partial}{\partial r}|n\ell m\rangle$

$= 0$ since $\hat{H}|n\ell m\rangle = E_n|n\ell m\rangle$

$\langle n\ell m|\hat{H} = \langle n\ell m| E_n$

Now $\left\langle\left[\frac{\partial}{\partial r}, \hat{H}\right]\right\rangle = \frac{\hbar^2 \ell(\ell+1)}{2m}\left\langle\left[\frac{\partial}{\partial r}, \frac{1}{r^2}\right]\right\rangle - Ze^2\left\langle\left[\frac{\partial}{\partial r}, \frac{1}{r}\right]\right\rangle$

Now $\left[\frac{\partial}{\partial r}, \frac{1}{r^2}\right]f = \frac{\partial}{\partial r}\left(\frac{f}{r^2}\right) - \frac{1}{r^2}\frac{\partial f}{\partial r} = \frac{-2f}{r^3}$

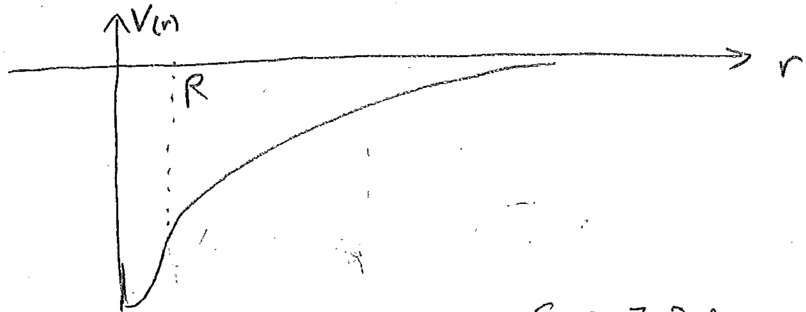
$\left[\frac{\partial}{\partial r}, \frac{1}{r}\right]f = \frac{\partial}{\partial r}\left(\frac{f}{r}\right) - \frac{1}{r}\frac{\partial f}{\partial r} = -\frac{f}{r^2}$

$$\Rightarrow \left\langle\left[\frac{\partial}{\partial r}, \hat{H}\right]\right\rangle = -\frac{\hbar^2 \ell(\ell+1)}{m_e}\left\langle\frac{1}{r^3}\right\rangle + Ze^2\left\langle\frac{1}{r^2}\right\rangle = 0$$

$$\Rightarrow \boxed{\left\langle\frac{1}{r^3}\right\rangle_{n\ell} = \frac{Ze^2 m_e}{\ell(\ell+1)\hbar^2}\left\langle\frac{1}{r^2}\right\rangle = \frac{Z}{a_0}\frac{1}{\ell(\ell+1)}\left\langle\frac{1}{r^2}\right\rangle_{n\ell}}$$

For a one dimensional atom with nucleus of charge Ze and radius R, the potential is

$$V(r) = \begin{cases} -\frac{3}{2}\frac{Ze^2}{R}\left(1-\frac{r^2}{3R^2}\right) & r<R \\ -\frac{Ze^2}{r} & r>R \end{cases}$$



(a) $H_1 = H_{finite} - H_{point} = \begin{cases} -\frac{3}{2}\frac{Ze^2}{R}\left(1-\frac{r^2}{3R^2}-\frac{2}{3}\frac{R}{r}\right) & r<R \\ 0 & r>R \end{cases}$

$$E_0^{(1)} = \langle 1s|H_1|s\rangle = -\frac{3}{2}\frac{Ze^2}{R}\int_0^R u_{10}^*\left(1-\frac{r^2}{2R^2}-\frac{2}{3}\frac{R}{r}\right)u_{10}\,dr$$

Scaling to the appropriate atomic units of length and energy

$$r \Rightarrow \left(\frac{a_0}{Z}\right)r \qquad E \rightarrow \left(\frac{Z^2e^2}{a_0}\right)E$$

$u(r) = 2re^{-r}$ (in these "atomic" units)

$$E_0^{(1)} = -\frac{6}{R}\int_0^R\left(r^2e^{-2r} - \frac{r^4}{3R^2}e^{-2r} - \frac{2}{3}Rre^{-2r}\right)dr$$

$$= -\frac{6}{R}\left[-\frac{e^{-2r}}{2}\left(r^2+r+\frac{1}{2}\right)\Big|_0^R + \frac{e^{-2r}}{6R^2}\left(r^4+r^3+3r^2+3r+\frac{3}{2}\right)\Big|_0^R - \frac{2R}{3}\left(-\frac{e^{-2r}}{2}\left(r+\frac{1}{2}\right)\right)\Big|_0^R\right]$$

$$= -\frac{6}{R}\left[e^{-2R}\left(\cancel{-\frac{1}{2}R^2}\cancel{-\frac{1}{2}R}\cancel{-\frac{1}{4}}+\cancel{\frac{1}{6}R^2}+\cancel{\frac{1}{3}R}+\cancel{\frac{1}{2}}+\frac{1}{2R}+\frac{1}{4R^2} + \cancel{\frac{1}{3}R^2}+\cancel{\frac{1}{6}R}\right) + \frac{1}{4} - \frac{R}{6} - \frac{1}{4R^2}\right]$$

$$\Rightarrow E_0^{(1)} = -\frac{6}{R}\left[\frac{e^{-2R}}{4R^2}\left[R^2+2R+1\right] + \frac{1}{4R^2}\left[R^2 - \frac{2R^3}{3} - 1\right]\right]$$

$$\boxed{E_0^{(1)} = -\frac{3}{2R^3}\left[e^{-2R}(R+1)^2 + R^2 - \frac{2}{3}R^3 - 1\right]}$$

or in usual CGS units $R \to R\left(\frac{Z}{a_0}\right)$

$E_0^{(1)} \to E_0^{(1)} \cdot \frac{Z^2e^2}{a_0}$

Equation (a)

(b) For Pb $Z=82$ $R = 8\times10^{-13}$ cm (Electronic state:)

$\bar{R} = R\left(\frac{Z}{a_0}\right) = 1.24\times10^{-2}$ $E_0^{(0)} = \frac{Z^2e^2}{2a_0} = 90.77$ keV

$$\Rightarrow \boxed{\frac{E_0^{(1)}}{E_0^{(0)}} = 2\times \text{Equation (a)} = 1.12\times10^{-4}}$$

$$\boxed{\frac{\langle r^{(0)}\rangle}{R} = \left(\frac{a_0}{Z}\right)\frac{1}{R} = 80.64}$$

- For the muon bound by the Lead Nucleus:

First we must rescale the Bohr radius and the Rydberg to to reduced mass of the two bodied system.

$m_{red} \approx m_\mu \cong 206.77\ m_e$

$a_0 \Rightarrow a_0\left(\frac{m_e}{m_{red}}\right) \cong \frac{a_0}{206.77}$

thus the dimensionless Radius we have defined

$\bar{R} = R\left(\frac{206.77}{a_0}\right)Z = 2.56$ $(R > \langle r\rangle)$

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The ground state of the "unperturbed" system

$$E_0^{(0)} = \frac{1}{2} \frac{Z^2 e^2}{a_0} (206.77) = 18.9 \text{ MeV}$$

$$\Rightarrow \frac{E_0^{(1)}}{E_0^{(0)}} = 2 \times \text{Equation (a)} (\bar{R})$$

[Again since the energy we found was in units of $\frac{Z^2 e^2}{a_0^*}$ and the ground state energy $= \frac{1}{2} \frac{Ze^2}{a_0^*}$ where a_0^* is the rescaled Bohr radius]

$$= +3 \frac{1}{(2.56)^3} \left[e^{-2(2.56)} (2.56+1)^2 + (2.56)^2 - \frac{2}{3}(2.56)^3 - 1 \right]$$

$$\boxed{\frac{E_0^{(1)}}{E_0^{(0)}} = 0.9945} \qquad \boxed{\frac{\langle r^{(0)} \rangle}{R} = \frac{a_0}{(206.7) Z} \frac{1}{R} = 0.39}$$

(c) To summarize what we calculated in part (b)

- For the bound electron the mean radius is 80 times larger than the radius of the nucleus, and thus, the electron spends "little time" inside the nucleus (or equivalently $\psi(\vec{x}) \ll 1$ for $r < R$). Thus, the corrections for finite nucleus radius is very small ($\sim 0.01\%$)

- For the bound muon, the mean radius of the unperturbed state is <u>smaller</u> than the nuclear radius, and thus the muon spends little time outside the nucleus. Therefore the correction for finite potential is not a perturbation, but the dominant effect. Thus, our result in part (b) is $\boxed{\text{unreliable}}$ as is reflected in the fact (over) that the first order correction were on the order of the zeroth order energy

A more accurrate model can be formulated by noting that inside the nucleus the potential in parabolic.

Thus we can model the muonic atom as being a muon bound in an isotropic harmonic oscillator potential. (Note: this was the Thompson model of the atom - "plumb pudding")

Thus $V_{eff} = -\frac{3}{2}\frac{Ze^2}{R} + \frac{1}{2}\left(\frac{Ze^2}{R^3}\right) r^2$

$= V_0 + \frac{1}{2} k r^2$ where $V_0 = -\frac{3}{2}\frac{Ze^2}{R}$, $k = \frac{Ze^2}{R^3}$

The Ground State energy is thus

$$E_{ground} - V_0 = \frac{3}{2}\hbar\omega_0 = \frac{3}{2}\hbar\sqrt{\frac{k}{M_\mu}} = \frac{3}{2}\hbar\sqrt{\frac{Ze^2}{M_\mu R^3}} = 13.843 \text{ MeV}$$

$$\boxed{E_{ground} = -8.3 \text{ MeV}}$$

Also the mean radius for an istropic harmonic oscillator:

$$\langle r\rangle = 3\sqrt{\langle x^2\rangle} = \frac{3}{2}\sqrt{\frac{\hbar}{2m_\mu\omega_0}} = 6.71\times10^{-13}\text{ cm}$$

$$\Rightarrow \boxed{\frac{\langle r\rangle}{R} = 0.839}$$

Thus $\langle r\rangle < R$ as expected, but not as small as predicted by a Bohr model.