Problems 1: Landé Projection Theorem

The Landé Projection Theorem (LPT) is a special case of the Wigner-Eckart Theorem for the case that the initial and final state are the same angular momentum. It states:

\[ \langle \alpha'; j m' | \hat{V} | \alpha; j m \rangle = \frac{\langle \alpha'; j | \hat{J} \cdot \hat{V} | \alpha; j \rangle}{j(j+1)} \langle j m' | \hat{J} | j m \rangle, \]

where \( \hat{V} \) is a vector operator w.r.t. \( \hat{J} \).

(a) Give a geometric interpretation of this in terms of a vector picture.

(b) To prove this theorem, take the following steps (do not give verbatim, Sakurai’s derivation):

(i) Show that \( \langle \alpha'; j, m | \hat{J} \cdot \hat{V} | \alpha; j, m \rangle = \langle \alpha'; j | \hat{J} \hat{J} | \alpha; j \rangle \), independent of \( m \).

(ii) Use this to show, \( \langle \alpha; j | \hat{J} | \alpha; j \rangle^2 = j(j+1) \) independent of \( \alpha \).

(iii) Show that \( \langle j m' | q | j m \rangle = \langle j m' | \hat{J} | j m \rangle / \sqrt{j(j+1)} \).

(iv) Put it all together to prove the LPT.

(c) As an application of the LPT, consider the Zeeman interaction for hydrogen.

\[ \hat{H}_{\text{int}} = -\hat{\mu} \cdot B, \]

where the magnetic dipole operators is \( \hat{\mu} = -\mu_B (\hat{L} + 2\hat{S}) \), ignoring the nuclear magnetic moment.

If the Zeeman effect is small compared to the fine-structure, but ignoring hyperfine structure, use the LPT to show that in a state \( nL_J \), the magnetic moment has the form,

\[ \hat{\mu} = -g_J \mu_B \hat{J}, \]

where \( g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)} \) is known as the Landé g-factor.

\[ \text{Hint: Use } J \cdot L = L^2 + \frac{1}{2}(J^2 - L^2 - S^2) \text{ and } J \cdot S = S^2 + \frac{1}{2}(J^2 - L^2 - S^2). \]

(d) Estimate the magnetic field strength at which the Zeeman interaction becomes on the order of the fine structure splitting between the \( 2p_{1/2} \) and \( 2p_{3/2} \) state in hydrogen.

(e) Now solve for the energy shift of the sublevels exactly, including both the spin-orbit and Zeeman interaction, as in P.S.02, #3. Show that for small \( B \) fields, one arrives at the “linear Zeeman shift” found in part (c).
Problem 2: Natural lifetimes of Hydrogen (10 points)

Though in the absence of any perturbation, an atom in the excited state will stay there forever (it is a stationary state), in reality it will “spontaneously decay” to the ground state. Fundamentally this occurs because the atom is always perturbed by “vacuum fluctuations” in the electro-magnetic field. The spontaneous emission rate on a dipole allowed transition from initial excited state $\psi_e$ to all allowed grounds states $\psi_g$ is,

$$\Gamma = \frac{4}{3h} k^3 \sum_g \left| \langle \psi_g | d | \psi_e \rangle \right|^2,$$

where $k = \omega_{eg} / c$ is the emitted photon’s wave number.

Consider now hydrogen including fine-structure. For a given sublevel, the spontaneous emission rate is

$$\Gamma_{(nLM,J)} \rightarrow (n' L' J') = \frac{4}{3h} k^3 \sum_{M_J} \left| \langle n'L'J'M' | d | nLMJ \rangle \right|^2.$$

(a) Show that the spontaneous emission rate is independent of the initial $M_J$. Explain this result physically.

(b) Calculate the lifetime ($\tau = 1/\Gamma$) of the $2p_{1/2}$ state in seconds.

Problem 3: Three spherical harmonics

As we have seen many times, often we need to calculate an integral of the form,

$$\int d\Omega Y^*_{l_3 m_3} (\theta, \phi) Y_{l_2 m_2} (\theta, \phi) Y_{l_1 m_1} (\theta, \phi).$$

This can be interpreted as the matrix element $\langle l_3 m_3 | \hat{Y}^{(l_2)}_{m_2} | l_1 m_1 \rangle$, where $\hat{Y}^{(l_2)}_{m_2}$ is an irreducible tensor operator.

(a) Use the Wigner-Eckart theorem to determine the restrictions on the quantum numbers so that this integral does not vanish.

(b) Given the “addition rule” for Legendre polynomials:

$$P_{1_3} (\mu) P_{1_2} (\mu) = \sum_{1_0} \langle l_3 0 | l_0 0 \rangle^2 P_{1_3} (\mu),$$

Use the Wigner-Eckart theorem to prove

$$\int d\Omega Y^*_{l_1 m_1} (\theta, \phi) Y_{l_2 m_2} (\theta, \phi) Y_{l_3 m_3} (\theta, \phi) = \frac{(2l_2 + 1)(2l_1 + 1)}{4\pi(2l_3 + 1)} \langle l_3 0 | l_2 0 | l_1 0 \rangle \langle l_3 m_3 | l_2 m_2 | l_1 m_1 \rangle$$

Hint: Consider $\langle l_3 0 | \hat{Y}^{(l_2)}_{m_2} | l_1 0 \rangle$