Physics 531: Atomic Physics

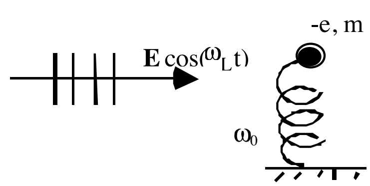
Problem Set #5

Due Wednesday, November 2, 2011

Problem 1: The ac-Stark effect

Suppose an atom is perturbed by a monochromatic electric field oscillating at frequency ω_L $\mathbf{E}(t) = E_z \cos(\omega_L t)\mathbf{e}_z$ (such as from a linearly polarized laser), rather than the dc-field studied in class. We know that such field can be absorbed and cause transitions between the energy levels; we will systematically study this effect later in the semester. The laser will also cause a *shift* of energy levels of the unperturbed states, known alternatively as the "ac-Stark shift", the "light shift", and sometimes the "Lamp shift" (don't you love physics humor). In this problem, we will look at this phenomenon is the simplest case that the field is near to resonance between the ground state $|g\rangle$ and some excited state $|e\rangle$, $\omega_L \approx \omega_{eg} \equiv \left(E_e - E_g\right)/\hbar$, so that we can ignore all other energy levels in the problem (the "two-level atom" approximation).

(i) The classical picture. Consider first the "Lorentz oscillator" model of the atom – a charge on a spring – with natural resonance ω_0.



The Hamiltonian for the system is $H = \dfrac{p^2}{2m} + \dfrac{1}{2} m\omega_0^2 z^2 - \mathbf{d}\cdot\mathbf{E}(t)$, where $d = -ez$ is the dipole.

(a) Ignoring damping of the oscillator, use Newton's Law to show that the induced dipole moment is

$$\mathbf{d}_{induced}(t) = \alpha\mathbf{E}(t) = \alpha E_z \cos(\omega_L t),$$

$$\text{where } \alpha = \frac{e^2/m}{\omega_0^2 - \omega_L^2} \approx \frac{-e^2}{2m\omega_0\Delta} \text{ is the polarizability with } \Delta \equiv \omega_L - \omega_0 \text{ the "detuning".}$$

(b) Use your solution to show that the total energy store in the system is $\overline{H} = -\dfrac{1}{4}\alpha E_z^2$.

Note, the factor of 1/4 arise because energy is required to *create* the dipole and we are taking the time average.

(ii) Quantum picture. We consider the two-level atom described above. The Hamiltonian for this system can be written in a time independent form (equivalent to the time-averaging done in the classical case)

$$\hat{H} = \hat{H}_{atom} + \hat{H}_{\text{int}},$$

where $\hat{H}_{atom} = -\hbar\Delta|e\rangle\langle e|$ is the "unperturbed" atomic Hamiltonian, and

$\hat{H}_{\text{int}} = -\frac{\hbar\Omega}{2}\left(|e\rangle\langle g| + |g\rangle\langle e|\right)$ is the dipole-interaction with $\hbar\Omega \equiv \langle e|\mathbf{d}|g\rangle \cdot \mathbf{E}$.

(a) Find the *exact* energy eigenvalues and eigenvectors for this simple two dimensional Hilbert space and plot the levels as a function of Δ. These are known as the atomic "dressed states".

(b) Expand your solution in (a) to lowest nonvanishing order in Ω to find the perturbation to the energy levels. Under what condition is this expansion valid?

(c) Confirm your answer to (b) using perturbation theory. Find also the mean induced dipole moment (to lowest order in perturbation theory), and from this show that the atomic polarizability, defined by $\langle \mathbf{d} \rangle = \alpha \mathbf{E}$ is $\alpha = \frac{-|\langle e|\mathbf{d}|g\rangle|^2}{\hbar\Delta}$, so that the second order perturbation to the ground state is $E_g^{(2)} = -\frac{1}{4}\alpha E_z^2$ as in part (b).

(d) Show that the ratio of the polarizability calculated classical in (b) and the quantum expression in (c) has the form

$$f \equiv \frac{\alpha_{quantum}}{\alpha_{classical}} = \frac{|\langle e|z|g\rangle|^2}{\left(\Delta z^2\right)_{SHO}},$$ where $\left(\Delta z^2\right)_{SHO}$ the SHO zero point variance.

This ratio is known as the oscillator strength.

Lessons:

- In lowest order perturbation theory an atomic resonance look just like a harmonic oscillator, with a correction factor given by the oscillator strength.
- Off-resonance harmonic perturbations cause energy level shifts as well as absorption and emission.

Problem 2: Light-shift for multilevel atoms

We found the AC-Stark (light shift) for the case of a two-level atom driven by a monochromatic field. In this problem we want to look at this phenomenon in a more general context, including arbitrary polarization of the electric field, and atoms with multiple sublevels.

Consider then a general monochromatic electric field $\mathbf{E}(\mathbf{x},t) = \mathrm{Re}\left(\mathbf{E}(\mathbf{x})e^{-i\omega_L t}\right)$, driving an atom near resonance on the transition, $\left|g;J_g\right\rangle \rightarrow \left|e;J_e\right\rangle$, where the ground and excited manifolds are each described by some total angular momentum J with degeneracy $2J+1$. The generalization of the AC-Stark shift is now the light-shift operator acting on the $2J_g+1$ dimensional ground manifold:

$$\hat{V}_{LS}(\mathbf{x}) = -\frac{1}{4}\mathbf{E}^*(\mathbf{x})\cdot\hat{\overleftrightarrow{\alpha}}\cdot\mathbf{E}(\mathbf{x}).$$

Here $\hat{\overleftrightarrow{\alpha}} = -\dfrac{\hat{\mathbf{d}}_{ge}\hat{\mathbf{d}}_{eg}}{\hbar\Delta}$ is the atomic polarizability tensor operator, where $\hat{\mathbf{d}}_{eg} \equiv \hat{P}_e\hat{\mathbf{d}}\hat{P}_g$ is the dipole operator, projected between the ground and excited manifolds; the projector onto the excited manifold is, $\hat{P}_e = \sum_{M_e=-J_e}^{J_e}\left|e;J_e,M_e\right\rangle\left\langle e;J_e,M_e\right|$, and similarly for the ground.

(a) By expanding the dipole operator in the spherical basis, **show** that the polarizability operator can be written,

$$\hat{\overleftrightarrow{\alpha}} = \tilde{\alpha}\left(\sum_{q,M_g}\left|C_{M_g}^{M_g+q}\right|^2 \vec{\mathbf{e}}_q\left|g;J_g,M_g\right\rangle\left\langle g;J_g,M_g\right|\vec{\mathbf{e}}_q^* + \sum_{q\neq q',M_g} C_{M_g+q-q'}^{M_g+q}\,C_{M_g}^{M_g+q}\;\vec{\mathbf{e}}_{q'}\left|g;J_g,M_g+q-q'\right\rangle\left\langle g;J_g,M_g\right|\vec{\mathbf{e}}_q^*\right),$$

$$\text{where } \tilde{\alpha} \equiv -\frac{\left|\left\langle e;J_e\|d\|g;J_g\right\rangle\right|^2}{\hbar\Delta} \text{ and } C_{M_g}^{M_e} \equiv \left\langle J_e\,M_e\middle|1q\,J_g\,M_g\right\rangle.$$

Explain physically, using dipole selection rules, the meaning of the expression for $\hat{\overleftrightarrow{\alpha}}$.

(b) Consider a polarized plane wave, with complex amplitude of the form, $\mathbf{E}(\mathbf{x}) = E_1\,\vec{\varepsilon}_L\,e^{i\mathbf{k}\cdot\mathbf{x}}$ where E_1 is the amplitude and $\vec{\varepsilon}_L$ the polarization (possibly complex). For an atom driven on the transition $\left|g;J_g=1\right\rangle \rightarrow \left|e;J_e=2\right\rangle$ and the cases (i) linear polarization along z, (ii) positive helicity polarization, (iii) linear polarization along x, **find** the eigenvalues and eigenvectors of the light-shift operator. Express the eigenvalues in units of $V_1 = -\frac{1}{4}\tilde{\alpha}\left|E_1\right|^2$. Please ***comment*** on what you find for cases (i) and (iii). **Repeat** for $\left|g;J_g=1/2\right\rangle \rightarrow \left|e;J_e=3/2\right\rangle$ and **comment**.

(c) A deeper insight into the light-shift potential can be seen by expressing the polarizability operator in terms of irreducible tensors. Verify that the total light shift is the sum of scalar, vector, and rank-2 irreducible tensor interaction,

$$\hat{V}_{LS} = -\frac{1}{4}\left(|\mathbf{E}(\mathbf{x})|^2 \hat{\alpha}^{(0)} + (\mathbf{E}^*(\mathbf{x}) \times \mathbf{E}(\mathbf{x})) \cdot \hat{\alpha}^{(1)} + \mathbf{E}^*(\mathbf{x}) \cdot \hat{\alpha}^{(2)} \cdot \mathbf{E}(\mathbf{x})\right),$$

$$\text{where } \hat{\alpha}^{(0)} = \frac{\hat{\mathbf{d}}_{ge} \cdot \hat{\mathbf{d}}_{eg}}{-3\hbar\Delta},\ \hat{\alpha}^{(1)} = \frac{\hat{\mathbf{d}}_{ge} \times \hat{\mathbf{d}}_{eg}}{-2\hbar\Delta},\ \hat{\alpha}^{(2)}_{ij} = \frac{\hat{\mathbf{d}}^i_{ge}\hat{\mathbf{d}}^j_{eg} + \hat{\mathbf{d}}^i_{ge}\hat{\mathbf{d}}^j_{eg}}{-2\hbar\Delta} - \hat{\alpha}^{(0)}\delta_{ij}.$$

(d) For the particular case of $|g; J_g = 1/2\rangle \to |e; J_e = 3/2\rangle$, **show** that the rank-2 tensor part *vanishes*. **Show** that the light-shift operator can be written in a basis independent *form* of a scalar interaction (independent of the sublevel), plus an effective Zeeman interaction for a fictitious B-field interacting with the spin 1/2 ground state,

$$\hat{V}_{LS} = V_0(\mathbf{x})\hat{1} + \mathbf{B}_{fict}(\mathbf{x}) \cdot \hat{\vec{\sigma}}$$

where

$$V_0(\mathbf{x}) = \frac{2}{3}V_1|\vec{\varepsilon}_L(\mathbf{x})|^2 \quad \text{(proportional to field intensity) and}$$

$$\mathbf{B}_{fict}(\mathbf{x}) = \frac{1}{3}V_1\left(\frac{\vec{\varepsilon}_L^*(\mathbf{x}) \times \vec{\varepsilon}_L(\mathbf{x})}{i}\right), \text{(proportional to the field ellipticity),}$$

and I have written $E(\mathbf{x}) = E_1\vec{\varepsilon}_L(\mathbf{x})$. Use this form to **explain** your results form part (b) on the transition $|g; J_g = 1/2\rangle \to |e; J_e = 3/2\rangle$.

(e) **Extra Credit:** For $|g; J_g = 1/2\rangle \to |e; J_e = 3/2\rangle$, explicitly show that

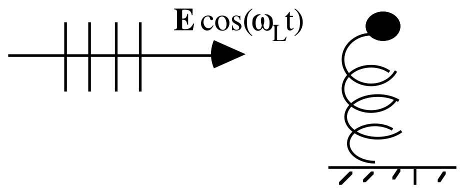
$$V_0(\mathbf{x}) = \frac{2}{3}V_1|\vec{\varepsilon}_L(\mathbf{x})|^2 \quad \text{(proportional to field intensity) and}$$

$$\mathbf{B}_{fict}(\mathbf{x}) = \frac{1}{3}V_1\left(\frac{\vec{\varepsilon}_L^*(\mathbf{x}) \times \vec{\varepsilon}_L(\mathbf{x})}{i}\right), \text{(proportional to the field ellipticity),}$$

and I have written $E(\mathbf{x}) = E_1\vec{\varepsilon}_L(\mathbf{x})$. Use this form to **explain** your results from part (b) on the transition $|g; J_g = 1/2\rangle \to |e; J_e = 3/2\rangle$.

Problem 3: Lorentz Classical Model of Absorption and Emission

Suppose we were to model an atom as an electron on a spring - i.e. a damped simple harmonic oscillator of mass m, with resonance frequency ω_0, and damping constant Γ. Consider driving the oscillator with a monochromatic plane wave, of frequency ω_L.



(a) Show that rate at which the dipole absorbs energy from the field, given by the rate at which the field does work on the charge averaged over one period, is

$$\frac{dW_{abs}}{dt} = \frac{\pi e^2 |\mathbf{E}|^2}{4m} g(\omega_L), \text{ where } g(\omega) = \frac{\Gamma/(2\pi)}{(\omega - \omega_{eg})^2 + \Gamma^2/4} \text{ is the line shape.}$$

Assume near resonance so that $\Delta = \omega_L - \omega_0 << \omega_L, \omega_0$.

(b) The absorption cross section, σ_{abs}, is defined as the rate at which energy is absorbed by an atom, divided by the flux, Φ, of photons incident on the atom, $\Phi \equiv I/\hbar\omega_L$ (i.e. the rate of photons incident on the atom per unit area), where $I = \frac{c}{8\pi}|\mathbf{E}_0|^2$ is the incident intensity (CGS units). Show that the classical model of absorption gives,

$$\sigma_{classical} = \frac{2\pi^2 e^2}{mc} g(\omega_L),$$

Evaluate this on-resonance, for a the parameters associated with Na, where the excitation wavelength is 589 nm and the linewidth (Full width at half-maximum) is 10 MHz.
The ratio of the integrated cross section an atomic transition and that given by the classical model to the quantum mechanical expression with equal resonance frequency and line width is known as the *oscillator* strength of the transition.

(c) From standard texts we have $\sigma_{quantum} = 4\pi^2 \frac{e^2}{\hbar c} |\langle e|\mathbf{x}|g\rangle|^2 \omega_L\, g(\omega_L)$, where $\langle e|\mathbf{x}|g\rangle$ is the matrix element of the electron position relative to the nucleus for the resonant transition. **Show** that on resonance,

$$f = \frac{\sigma_{quantum}}{\sigma_{classical}} = \frac{2m\omega_0}{\hbar} |\langle e|x|g\rangle|^2.$$

Note that $\hbar / 2m\omega_0$ is the square of the characteristic length scale of a quantum simple harmonic oscillator . Thus, the oscillator strength measures the ratio of dipole oscillation amplitude for a two level atom as compared to a simple harmonic oscillator.

Let us now assume that our spring has no *intrinsic* damping associated with it. Consider the scattering of an electromagnetic wave by this oscillating charge. As the charge radiates, the electromagnetic field will carry away energy. This energy must come from the kinetic energy of the accelerating charge. Thus the very act of radiating should "damp" the motion of the charge. This is known as radiation reaction, and will determine a classical decay rate Γ_{class} for the oscillator. In steady state the power radiated by the charge (given by the classical Larmor formula) is equal to the power absorbed.

(d) Assume that the oscillator is damped as Γ_{class}, and show that

$P_{abs} = P_{radiated} \Rightarrow \Gamma_{class} = \frac{2}{3}\frac{e^2}{mc^3}\omega^2 = \frac{2}{3}\,k r_c\,\omega$,where r_c is the classical electron radius.

(e) Show that the quantum mechanical decay rate is related to the classical formula by

$$\Gamma_{quantum} = f\,\Gamma_{class},\quad \text{where } f \text{ is the oscillator strength.}$$