Physics 531 – Lecture 6 – Hyperfine Structure

So far we have neglected effects of the nucleus in our description of the spectrum of hydrogen. Generally, these effects are very small, even smaller than the relativistic corrections that lead to the "fine-structure" in the spectrum. As such these effects lead the "hyperfine-structure" in the spectrum. Examples include

(i) Reduced mass correction

(ii) Finite nuclear volume correction

(iii) Field of nucleus beyond electro-static coulomb.

The first two effects do not change the basic interaction, central coulomb potential, and as such, they do not change the overall level structure; the quantum numbers remain unchanged. Since they vary from nucleus to nucleus of the same atomic number, they give rise to what is known as an isotope shift.

Effect (iii) arises from (to lowest order)

- Magnetic dipole field of nucleus due to its intrinsic (spin) magnetic moment.
- Electric quadrupole field of nucleus due to its nonspherical shape

These latter two break the symmetry of the coulomb potential & add different structure (new quantum numbers) to the energy levels.

Generally, the electric quadrapole effect is smaller than the magnetic dipole, so we will concentrate on the latter

<u>Magnetic dipole hyperfine structure</u>

The nucleus of an atom general has intrinsic spin angular momentum, usually denoted by the operator $\hat{\vec{I}}$. This come along with an intrinsic magnetic moment $\hat{\vec{\mu}}_N = \gamma_N \hat{\vec{I}}$ where γ_N is the "gyromagnetic ratio". Determining the value of γ_N from first principles is difficult, since the nucleons are constructed from the elementary quarks and qcd calculations are not usually possible. We determine this empirically. For the proton

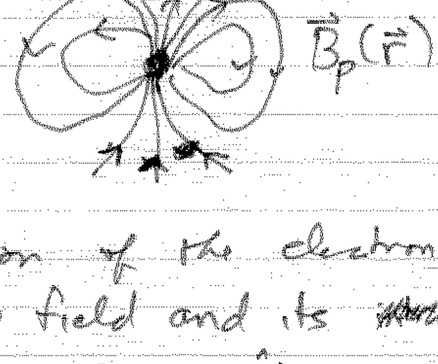
$$\hat{\vec{\mu}}_p = g_p \mu_N \hat{\vec{I}} \quad (\hat{\vec{I}} \text{ in units of } \hbar)$$

where $\mu_N = \frac{e\hbar}{2M_p c}$ $\quad$ (M_p = mass proton)

nuclear magneton

and $g_p \cong 5.6$ is the proton's g-factor

The proton (generally nucleus) thus is a source of a magnetic dipole field



At the position of the electron, there is an interaction between this field and its magnetic moment

$$\vec{\mu}_e = -(g_e \mu_B \hat{\vec{S}} + \mu_B \hat{\vec{L}})$$

$$\mu_B = \frac{e\hbar}{2m_e c} \quad \text{(Bohr magneton)}$$

$$g_e \cong 2$$

$\Rightarrow$ Hyperfine interaction $\hat{H}_{HF} = -\hat{\vec{\mu}}_e \cdot \vec{B}_p(\vec{r})$

For an electron away from the origin, this is the dipole - dipole interaction energy

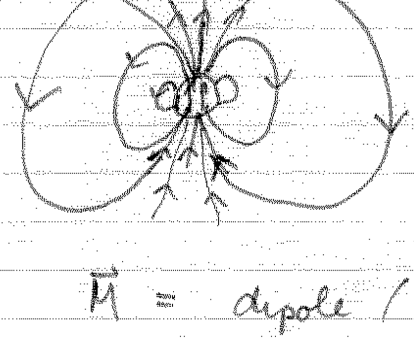
$$\hat{H}_{Dipole} = \frac{\hat{\vec{\mu}}_e \cdot \hat{\vec{\mu}}_p - 3(\hat{\vec{\mu}}_e \cdot \vec{e}_r)(\hat{\vec{\mu}}_p \cdot \vec{e}_r)}{r^3}$$

Note energy scale: $\frac{\mu_B \mu_N}{a_0^3} = \left(\frac{m_e}{M_p}\right) \frac{\mu_B^2}{a_0^3}$

Now $\mu_B \cong \alpha e a_0 \Rightarrow E_{HF} \sim \left(\frac{m_e}{M_p}\right) \alpha^2 \left(\frac{e^2}{a_0}\right)$ ← Hartree

Thus, the energy ~~scale~~ scale is reduced from fine-structure by the factor $\frac{m_e}{M_p} \sim \frac{1}{2000}$

Now ~~H_HF~~ at the origin we must be careful. The nucleus is actually a finite radius object, and thus the field is different near the origin



For a magnetized sphere, the field is

$$\vec{B}_{inside} = \frac{4\pi}{3} \vec{M} \quad \text{inside sphere}$$

where $\vec{M}$ = dipole / volume

Shrinking to a point $\Rightarrow \vec{B}_{inside} = \frac{8\pi}{3} \vec{\mu}_p \, \delta^{(3)}(\vec{r})$

Thus, there is an additional "contact term"

$$\hat{H}_{contact} = -\frac{8\pi}{3} \vec{\mu}_e \cdot \vec{\mu}_p \, \delta^{(3)}(\vec{r})$$

Finally, the total magnetic hyperfine interaction is

$$\hat{H}_{HF} = (-2g_N)\mu_B\mu_N \left[\left(\frac{\vec{L}\cdot\vec{I} + \vec{S}\cdot\vec{I} - 3(\vec{S}\cdot\vec{e}_r)(\vec{I}\cdot\vec{e}_r)}{r^3}\right) - \frac{8\pi}{3}\vec{S}\cdot\vec{I}\,\delta^{(3)}(\vec{r})\right]$$

Consider, first, the perturbation to the ground state.

Including electron and nuclear spin, there are four degenerate sublevels:

$n = 1 \quad l = 0 \quad m_s = \pm 1/2, \quad m_i = \pm 1/2$

$|1s\rangle \otimes \{|\uparrow\rangle, |\downarrow\rangle\}_{electron} \otimes \{|\Uparrow\rangle, |\Downarrow\rangle\}_{spin}$

Since $l=0$ the $\vec{L}\cdot\vec{I} \to 0$

Furthermore, the dipole term vanishes when average over the isotropic s-state for $l=0$

$\Rightarrow$ In Hydrogen ground state

$$l=0: \quad \hat{H}_{HF} = \left(\frac{16\pi}{3}g_N\right)\mu_B\mu_N\,\hat{\vec{I}}\cdot\hat{\vec{S}}\;\delta^{(3)}(\vec{r})$$

We can automatically diagonalize this Hamiltonian in the 4D subspace by going to the coupled representation of electron and nuclear spin, as we did for spin-orbit coupling.

Let $\vec{F} = \vec{I} + \vec{S} \Rightarrow \vec{F}^2 = \vec{I}^2 + \vec{S}^2 + 2\vec{I}\cdot\vec{S}$

$$\Rightarrow \vec{I}\cdot\vec{S} = \frac{1}{2}\left(\vec{F}^2 - \vec{I}^2 - \vec{S}^2\right)$$

For two spin ½ $\quad I = 1/2 \quad S = 1/2$

$\Rightarrow$ two possible totals $F = 0, 1$

Coupled states $|F M_F\rangle \quad (I = 1/2,\ S = 1/2)$

$\Rightarrow$ Eigenstates $|1s\rangle \otimes |F M_F\rangle$ ← Hyperfine sublevel

Eigenvalue perturbation

$$E_F^{(1)} = A \langle F M_F | \vec{I}\cdot\vec{S} | F M_F \rangle$$

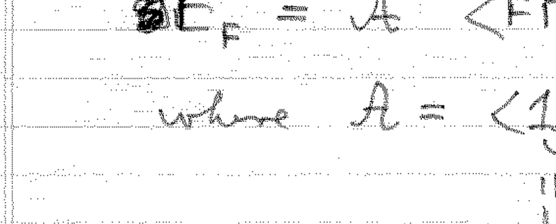
where $A = \langle 1s | \delta^{(3)}(\vec{r}) | 1s \rangle \frac{16\pi}{3} g_N \mu_B \mu_N$

$$\langle 1s | \delta^{(3)}(\vec{r}) | 1s \rangle = |\psi_{100}(0)|^2 = \frac{1}{4\pi} |R_{10}(0)|^2 = \frac{1}{a_0^3 \pi}$$

$$\Rightarrow A = \frac{16}{3} g_N \frac{\mu_B \mu_N}{a_0^3}$$

$$E_F^{(1)} = A \left[\frac{F(F+1) - 3/2}{2}\right] \rightarrow \quad E_{F=0}^{(1)} = -\frac{3}{4} A$$

$$E_{F=1}^{(1)} = \frac{1}{4} A$$

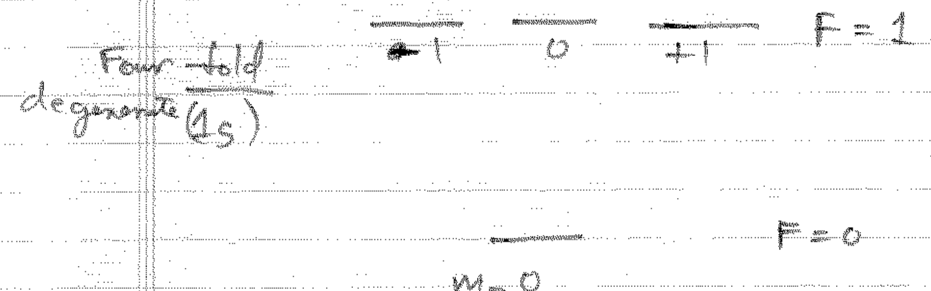


$\Delta E_{HF} = A = 1420\ \text{MHz} = 21\ \text{cm}$

$= 0.047\ \text{cm}^{-1}$

- Use in hydrogen maser
- Astrophysics $\Rightarrow$ Radio ~~Astronomy~~ Astronomy

With sublevels



Hyperfine in excited states (general $\vec{I}$)

Generally $\vec{F} = \vec{I} + \vec{J}$, $\vec{J} = \vec{L} + \vec{S}$

Example: $2p_{3/2} \Rightarrow l = 1 \quad j = 3/2 \quad i = 1/2$

$\Rightarrow f = 1, 2$

$2p_{1/2} \Rightarrow l = 1 \quad j = 1/2 \quad i = 1/2$

$f = 0, 1$