Problem 1: Landé Projection Theorem

The Landé Projection Theorem (LPT) is a special case of the Wigner-Eckart Theorem for the case that the initial and final state are the same angular momentum. It states:

\[
\langle \alpha'; j m' | \hat{\mathbf{V}} | \alpha; j m \rangle = \frac{\langle \alpha'; j | \hat{\mathbf{J}} \cdot \hat{\mathbf{V}} | \alpha; j \rangle}{j(j+1)} \langle j m' | \hat{\mathbf{j}} | j m \rangle,
\]

where \( \hat{\mathbf{V}} \) is a vector operator w.r.t. \( \hat{\mathbf{J}} \).

(a) Give a geometric interpretation of this in terms of a vector picture.

(b) To prove this theorem, take the following steps (do not give verbatim, Sakurai’s derivation):

(i) Show that \( \langle \alpha'; j, m | \hat{\mathbf{J}} \cdot \hat{\mathbf{V}} | \alpha; j, m \rangle = \langle \alpha'; j | \hat{\mathbf{J}} | \alpha; j \rangle \langle \alpha'; j | \hat{\mathbf{V}} | \alpha; j \rangle \), independent of \( m \).

(ii) Use this to show, \( \langle \alpha; j | \hat{\mathbf{J}} | \alpha; j \rangle \rangle = j(j+1) \) independent of \( \alpha \).

(iii) Show that \( \langle j m' | q jm \rangle = \langle j m' | V jm \rangle / \sqrt{j(j+1)} \).

(iv) Put it all together to prove the LPT.

(c) As an application of the LPT, consider the Zeeman interaction for hydrogen.

\[
\hat{H}_{\text{int}} = -\hat{\mu} \cdot \mathbf{B},
\]

where the magnetic dipole operators is \( \hat{\mu} = -\mu_b (g_s \hat{\mathbf{L}} + g_l \hat{\mathbf{S}}) \), with \( g_s = 1, g_l = 2 \).

If the Zeeman effect is small compared to the fine-structure, but ignoring hyperfine structure, use the LPT to show that in a state \( nL_j \), the magnetic moment has the form,

\[
\hat{\mu} = -g_J \mu_b \hat{\mathbf{j}}, \quad \text{where} \quad g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}
\]

is known as the Landé g-factor.

Hint: Use \( \mathbf{J} \cdot \mathbf{L} = \mathbf{L}^2 + \frac{1}{2}(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2) \) and \( \mathbf{J} \cdot \mathbf{S} = \mathbf{S}^2 + \frac{1}{2}(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2) \).

(d) Estimate the magnetic field strength at which the Zeeman interaction becomes on the order of the fine structure splitting between the \( 2p_{1/2} \) and \( 2p_{3/2} \) state in hydrogen.

(e) Now solve for the energy shift of the sublevels exactly, including both the spin-orbit and Zeeman interaction, as in P.S. #3. Show that for small \( B \) fields, one arrives at the “linear Zeeman shift” found in part (c).
Problem 2: Natural lifetimes of Hydrogen (10 points)

Though in the absence of any perturbation, an atom in the excited state will stay there forever (it is a stationary state), in reality it will “spontaneously decay” to the ground state. Fundamentally this occurs because the atom is always perturbed by ‘vacuum fluctuations’ in the electro-magnetic field. The spontaneous emission rate on a dipole allowed transition from initial excited state $|\psi_e\rangle$ to all allowed grounds states $|\psi_g\rangle$ is,

$$\Gamma = \frac{4}{3h} k^3 \sum_g \langle \psi_g | \hat{a} | \psi_e \rangle^2,$$

where $k = \omega_{eg} / c$ is the emitted photon’s wave number.

Consider now hydrogen including fine-structure. For a given sublevel, the spontaneous emission rate is

$$\Gamma_{(nLJM)\rightarrow(n'L'J')} = \frac{4}{3h} k^3 \sum_{M'_j} \langle n'L'J'M'_j | \hat{a} | nLJM_j \rangle^2.$$

(a) Show that the spontaneous emission rate is independent of the initial $M_J$. Explain this result physically.

(b) Calculate the lifetime ($\tau=1/\Gamma$) of the $2p_{1/2}$ state in seconds.

Problem 3: Three spherical harmonics

As we have seen many times, often we need to calculate an integral of the form,

$$\int d\Omega Y_{l_3m_3}^* (\theta, \phi) Y_{l_2m_2} (\theta, \phi) Y_{l_1m_1} (\theta, \phi).$$

This can be interpreted as the matrix element $\langle l_3m_3 | \hat{Y}_{m_2}^{(l_2)} | l_1m_1 \rangle$, where $\hat{Y}_{m_2}^{(l_2)}$ is an irreducible tensor operator.

(a) Use the Wigner-Eckart theorem to determine the restrictions on the quantum numbers so that this integral does not vanish.

(b) Given the “addition rule” for Legendre polynomials:

$$P_{l_1} (\mu) P_{l_2} (\mu) = \sum_{l_3} \langle l_30 | l_1l_20 \rangle^2 P_{l_3} (\mu),$$

Use the Wigner-Eckart theorem to prove

$$\int d\Omega Y_{l_3m_3}^* (\theta, \phi) Y_{l_2m_2} (\theta, \phi) Y_{l_1m_1} (\theta, \phi) = \sqrt{\frac{(2l_2+1)(2l_1+1)}{4\pi(2l_3+1)}} \langle l_30 | l_20l_10 \rangle \langle l_3m_3 | l_2m_2l_1m_1 \rangle$$

Hint: Consider $\langle l_30 | Y_{0}^{(l_3)} | l_10 \rangle$