Physics 531

Problem Set #3

Due Tuesday, Feb. 22, 2005

Problem 1: Landé Projection Theorem

The Landé Projection Theorem (LPT) is a special case of the Wigner-Eckart Theorem for the case that the initial and final state are the same angular momentum. It states:

$$\langle \alpha'; jm' | \hat{\mathbf{V}} | \alpha; jm \rangle = \frac{\langle \alpha'; j | \hat{\mathbf{J}} \cdot \hat{\mathbf{V}} | \alpha; j \rangle}{j(j+1)} \langle jm' | \hat{\mathbf{J}} | jm \rangle, \text{ where } \hat{\mathbf{V}} \text{ is a vector operator w.r.t. } \hat{\mathbf{J}}.$$

(a) Give a geometric interpretation of this in terms of a vector picture.
(b) To prove this theorem, take the following steps (do not give verbatim, Sakurai's derivation):

- (i) Show that $\langle \alpha'; j,m | \hat{\mathbf{J}} \cdot \hat{\mathbf{V}} | \alpha; j,m \rangle = \langle \alpha'; j \| J \| \alpha'; j \rangle \langle \alpha'; j \| V \| \alpha; j \rangle$, independent of *m*.
- (ii) Use this to show, $\left| \langle \alpha; j \| J \| \alpha; j \rangle \right|^2 = j(j+1)$ *independent of* α.
- (iii) Show that $\langle j m' | 1 q\, jm \rangle = \langle jm' | \hat{J}_q | jm \rangle / \sqrt{j(j+1)}$.
- (iv) Put it all together to prove the LPT.

(c) As an application of the LPT, consider the Zeeman interaction for hydrogen.

$$\hat{H}_{\text{int}} = -\hat{\vec{\mu}} \cdot \mathbf{B},$$

where the magnetic dipole operators is $\hat{\vec{\mu}} = -\mu_B (g_l \hat{\mathbf{L}} + g_s \hat{\mathbf{S}})$, with $g_l = 1,\ g_s = 2$.

If the Zeeman effect is small compared to the fine-structure, but ignoring hyperfine structure, use the LPT to show that in a state nL_J, the magnetic moment has the form,

$$\hat{\vec{\mu}} = -g_J \mu_B \hat{\vec{\mathbf{J}}}, \text{ where } g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)} \text{ is known as the Landé g-factor.}$$

$$\textit{Hint:} \text{ Use } \mathbf{J} \cdot \mathbf{L} = \mathbf{L}^2 + \frac{1}{2}\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right) \text{ and } \mathbf{J} \cdot \mathbf{S} = \mathbf{S}^2 + \frac{1}{2}\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right).$$

(d) Estimate the magnetic field strength at which the Zeeman interaction becomes on the order of the fine structure splitting between the $2p_{1/2}$ and $2p_{3/2}$ state in hydrogen.

(e) Now solve for the energy shift of the sublevels exactly, including both the spin-orbit and Zeeman interaction, as in P.S. #3. Show that for small **B** fields, one arrives at the "linear Zeeman shift" found in part (c).

Problem 2: Natural lifetimes of Hydrogen (10 points)

Though in the absence of any perturbation, an atom in the excited state will stay there forever (it is a stationary state), in reality it will "spontaneously decay" to the ground state. Fundamentally this occurs because the atom is always perturbed by 'vacuum fluctuations" in the electro-magnetic field. The spontaneous emission rate on a dipole allowed transition from initial excited state $\left|\psi_e\right\rangle$ to all allowed grounds states $\left|\psi_g\right\rangle$ is ,

$$\Gamma = \frac{4}{3\hbar}k^3 \sum_g \left|\left\langle\psi_g\right|\hat{\mathbf{d}}\left|\psi_e\right\rangle\right|^2, \text{ where } k = \omega_{eg}/c \text{ is the emitted photon's wave number.}$$

Consider now hydrogen including fine-structure. For a given sublevel, the spontaneous emission rate is

$$\Gamma_{(nLJM_J)\to(n'L'J')} = \frac{4}{3\hbar}k^3 \sum_{M_J'} \left|\left\langle n'L'J'M_J'\right|\hat{\mathbf{d}}\left|nLJM_J\right\rangle\right|^2 .$$

(a) Show that the spontaneous emission rate is *independent* of the initial M_J. Explain this result physically.

(b) Calculate the lifetime ($\tau=1/\Gamma$) of the $2p_{1/2}$ state in seconds.

Problem 3: Three spherical harmonics

As we have seen many times, often we need to calculate an integral of the form,

$$\int d\Omega\, Y^*_{l_3 m_3}(\theta,\phi)\, Y_{l_2 m_2}(\theta,\phi)\, Y_{l_1 m_1}(\theta,\phi) .$$

This can be interpreted as the matrix element $\left\langle l_3 m_3\right|\hat{Y}^{(l_2)}_{m_2}\left|l_1 m_1\right\rangle$, where $\hat{Y}^{(l_2)}_{m_2}$ is an irreducible tensor operator.

(a) Use the Wigner-Eckart theorem to determine the restrictions on the quantum numbers so that this integral does not vanish.

(b) Given the "addition rule" for Legendre polynomials:

$$P_{l_1}(\mu)\, P_{l_2}(\mu) = \sum_{l_3} \left\langle l_3 0\right|l_1 0 l_2 0\rangle^2 P_{l_3}(\mu) ,$$

Use the Wigner-Eckart theorem to prove

$$\int d\Omega\, Y^*_{l_3 m_3}(\theta,\phi)\, Y_{l_2 m_2}(\theta,\phi)\, Y_{l_1 m_1}(\theta,\phi) = \sqrt{\frac{(2l_2+1)(2l_1+1)}{4\pi(2l_3+1)}} \left\langle l_3 0\right|l_2 0 l_1 0\rangle \left\langle l_3 m_3\right|l_2 m_2 l_1 m_1\rangle$$

Hint: Consider $\left\langle l_3 0\right|\hat{Y}^{(l_2)}_0\left|l_1 0\right\rangle$