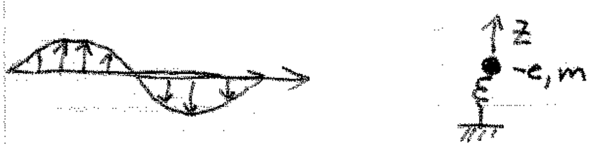
Physics 531 - Problem Set #5 : Solutions

Problem: The ac-Stark effect

Interaction of an induced oscillating dipole with an oscillating field $\vec{E} = \vec{E}_0 \cos\omega_L t \qquad \vec{\mathcal{E}}_0 = \mathcal{E}_z \vec{e}_z$

(i) Lorentz oscillator model:



(a) The incident field will drive oscillations of the charge at frequency ω_L. The eq. of motion

$$\ddot{z} + \omega_0^2 z = -\frac{e}{m}\mathcal{E}_z \cos\omega_L t$$

Go to complex amplitude $\quad z \equiv Re\left(Z_0 e^{-i\omega_L t}\right)$

$$\Rightarrow (-\omega_L^2 + \omega_0^2) Z_0 = -\frac{e}{m}\mathcal{E}_z$$

$$\Rightarrow \quad Z_0 = \left(\frac{-e/m}{\omega_0^2 - \omega_L^2}\right)\mathcal{E}_z$$

Induced dipole moment oscillating at drive-frequency ω_L

$$d_{induced}(t) = Re\left(-e Z_0 e^{-i\omega_L t}\right)$$

$$= \frac{+e^2/m}{\omega_0^2 - \omega_L^2}\mathcal{E}_z \cos\omega_L t$$

$$\Rightarrow \boxed{\vec{d}_{ind}(t) = \alpha\, \vec{\mathcal{E}}(t)} \qquad \boxed{\alpha = \frac{+e^2/m}{\omega_0^2 - \omega_L^2}}$$

(Next page)

In the near resonance approximation:

Let $\Delta \equiv \omega_L - \omega_0$ ("detuning") $\qquad \Delta << \omega_0 \sim |\omega_L|$

$$\Rightarrow \omega_0^2 - \omega_L^2 = (\omega_0 + \omega_L)(\omega_0 - \omega_L) = (2\omega_0 + \Delta)(-\Delta)$$

$$\approx -2\omega_0 \Delta \quad \text{(to first order in } \Delta << \omega_0)$$

$$\therefore \boxed{\alpha \approx \frac{-e^2}{2m\omega_0\Delta}}$$

(b) Total energy $\quad H = \frac{1}{2} m \dot{z}^2 + \frac{1}{2} m \omega_0^2 z^2 - \vec{d} \cdot \vec{E}$

$$\Rightarrow H = \frac{1}{2} m (\omega_0^2 - \omega_L^2) z^2 - \vec{d} \cdot \vec{E}$$

$$= \frac{1}{2} m (\omega_0^2 - \omega_L^2) \frac{e^2/m^2}{(\omega_0^2 - \omega_L^2)^2} E^2 - \alpha E^2$$

$$= \frac{1}{2} \alpha E^2 - \alpha E^2 = -\frac{1}{2} \alpha E^2$$

$$\Rightarrow \boxed{H = -\frac{1}{2} \alpha E^2(t) = -\frac{1}{2} \vec{d}_{ind}(t) \cdot \vec{E}(t)}$$

Time averaging $\quad \overline{\cos^2 \omega_L t} = 1/2$

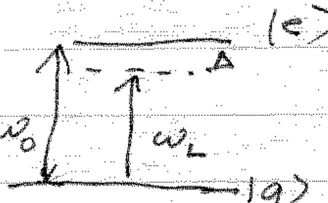
$$\Rightarrow \boxed{\overline{H} = -\frac{1}{4} \alpha E_2^2}$$

(ii) Quantum picture

(a) Given two level atom

$\Delta \ll \omega_0 \quad \Delta \ll |\omega_L|$

(ignore all other levels)



$\Rightarrow$ Effective Hamiltonian (we will derive later)

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

$$\hat{H}_0 = \hat{H}_{Atom} = -\hbar\Delta|e\rangle\langle e| \quad (\text{"unperturbed atom"})$$

$$\hat{H}_1 = \hat{H}_{int} = -\frac{\hbar\Omega}{2}\left(|e\rangle\langle g| + |g\rangle\langle e|\right) \quad (\text{"laser interaction"})$$

$$\Omega \equiv \frac{\langle e|\vec{d}|g\rangle \cdot \vec{E}}{\hbar} \quad \text{"Rabi frequency"}$$

(a) This simple 2-dimensional problem can be solve exactly: Matrix representation in bases $\{|e\rangle, |g\rangle\}$

$$H \doteq -\hbar\begin{bmatrix} \Delta & \Omega/2 \\ \frac{\Omega}{2} & 0 \end{bmatrix} = -\hbar\left(\frac{\Delta}{2}\hat{\mathbb{1}} + \frac{\Delta}{2}\hat{\sigma}_z + \frac{\Omega}{2}\hat{\sigma}_x\right)$$

$$= -\frac{\hbar\Delta}{2}\hat{\mathbb{1}} - \frac{\hbar\vec{\Omega}}{2}\cdot\vec{\sigma}$$

where $\vec{\Omega} = \Delta\vec{e}_z + \Omega\,\vec{e}_x$ (Generalized Rabi frequency)

$$\tilde{\Omega} \equiv |\vec{\Omega}| = \sqrt{\Omega^2 + \Delta^2} \qquad \frac{\vec{\Omega}}{\tilde{\Omega}} \equiv \cos\theta\,\vec{e}_z + \sin\theta\,\vec{e}_x$$

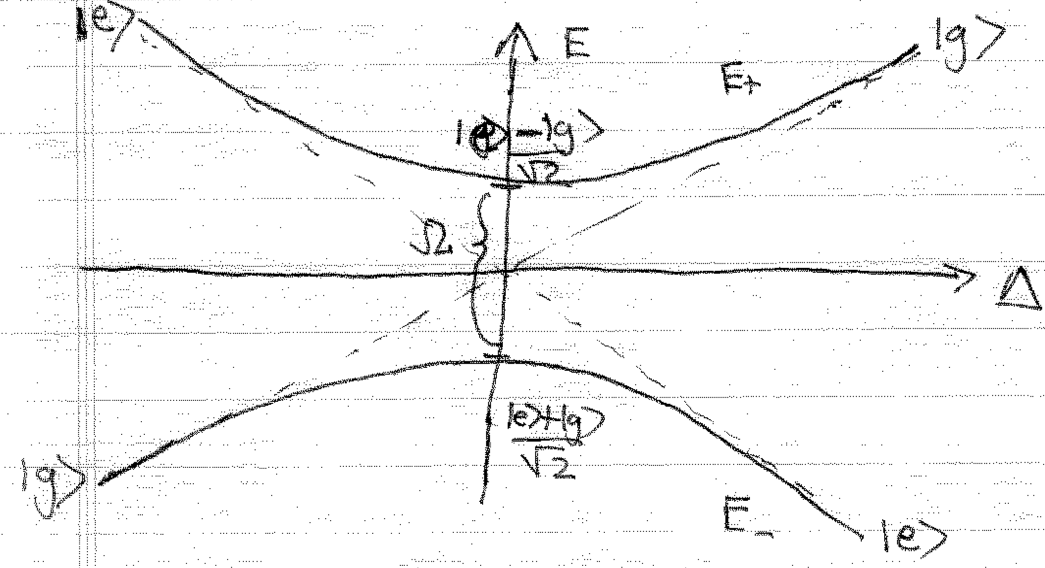
$$\tan\theta = \frac{\Omega}{\Delta}$$

$\Rightarrow$ Eigenvalues:

$$E_\pm = -\left(\frac{\hbar\Delta}{2} \pm \frac{\hbar\tilde{\Omega}}{2}\right) = -\frac{\hbar}{2}\left(\Delta \pm \sqrt{\Omega^2+\Delta^2}\right)$$

Eigenvectors: $|\pm\rangle = \cos\frac{\theta}{2}|e\rangle \pm \sin\frac{\theta}{2}|g\rangle$

$$\tan\theta = \frac{\Omega}{\Delta}$$



Typical anti-crossing behavior. At $\Delta=0$ $|e\rangle$ and $|g\rangle$ are "degenerate" in the absence of coupling. The field breaks the degeneracy into symmetric and anti-symmetric superpositions

Note for $\Delta<0$ ("red detuning") $|e\rangle$ is shifted up and $|g\rangle$ down (level repulsion)

For $\Delta>0$ ("blue detuning) the reverse occurs $\Rightarrow$ level attraction

Note: In class we saw that in static perturbation the levels always repell due to perturbation. This agrees with our result, since in the d.c. limit Δ is negative.

(b) Expansion for $\frac{\Omega}{\Delta} \ll 1$ $\left(\frac{\text{coupling matrix element}}{\text{energy level difference}} \right)$

$$E_{\pm} = -\frac{\hbar}{2}\left(\Delta \pm \Delta\sqrt{1+\frac{\Omega^2}{\Delta^2}}\right) \approx -\frac{\hbar}{2}\left(\Delta \pm \Delta\left(1+\frac{\Omega^2}{2\Delta^2}\right)\right)$$

$$\Rightarrow \boxed{E_+ \approx -\hbar\Delta \cdot \frac{\hbar\Omega^2}{4\Delta} \qquad E_- \approx -\frac{\hbar\Omega^2}{4\Delta}}$$

Lowest non-vanishing perturbation is second order in Ω

(c) Using perturbation theory $\hat{H}_1 = -\frac{\hbar\Omega}{2}\left(|e\rangle\langle g| + h.c.\right)$

0th order $E_e^{(0)} = -\hbar\Delta$ $\qquad E_g^{(0)} = 0$

$|e\rangle \qquad |g\rangle$

1st order $E_e^{(1)} = \langle e|\hat{H}_1|e\rangle = 0 \qquad E_g^{(1)} = \langle g|\hat{H}_1|g\rangle = 0$

2nd order
$$E_e^{(2)} = \frac{|\langle g|\hat{H}_1|e\rangle|^2}{E_e^{(0)} - E_g^{(0)}} = \frac{\frac{\hbar^2\Omega^2}{4}}{-\hbar\Delta} = -\frac{\hbar\Omega^2}{4\Delta}$$

$$E_g^{(2)} = \frac{|\langle e|\hat{H}_1|g\rangle|^2}{E_g^{(0)} - E_e^{(0)}} = \frac{\frac{\hbar\Omega^2}{4}}{\hbar\Delta} = \frac{\hbar\Omega^2}{4\Delta}$$

Thus to second order

$$E_e = E_e^{(0)} + E_e^{(2)} = -\hbar\Delta - \frac{\hbar\Omega^2}{4\Delta} \checkmark$$

as in (6)

$$E_g = E_g^{(0)} + E_g^{(2)} = +\frac{\hbar\Omega^2}{4\Delta} \checkmark$$

Mean dipole: (Assume atom starts in ground state)

To first order: $|\tilde{\phi}_g\rangle = \underset{\sim}{|g\rangle} + \frac{|e\rangle\langle e|\hat{H}_1|g\rangle}{E_g^{(0)} - E_e^{(1)}}$

$$= |\phi_g^{(0)}\rangle + |\phi_g^{(1)}\rangle$$

$$\Rightarrow |\tilde{\phi}_g\rangle = |g\rangle - \frac{\Omega}{2\Delta}|e\rangle \quad \text{(unnormalized)}$$

$$\langle \vec{d}\rangle = \frac{\langle\tilde{\phi}_g|\vec{d}|\tilde{\phi}_g\rangle}{\langle\tilde{\phi}_g|\tilde{\phi}_g\rangle} = \frac{-\frac{1}{2\Delta}\left(\Omega^*\langle e|\vec{d}|g\rangle + \Omega\langle g|\vec{d}|e\rangle\right)}{1 + \underbrace{\frac{\Omega^2}{4\Delta^2}}_{\text{neglect}}}$$

$\Rightarrow$ To lowest order in $\Omega = \frac{\langle e|\vec{d}|g\rangle \cdot \vec{\mathcal{E}}}{\hbar}$

$$\boxed{\langle\vec{d}\rangle = -\frac{|\langle e|\vec{d}|g\rangle|^2}{\hbar\Delta}\,\vec{\mathcal{E}} = \alpha\,\vec{\mathcal{E}}}$$

Now $E_g^{(2)} = \frac{\hbar\Omega^2}{4\Delta} = \frac{|\langle e|\vec{d}|g\rangle|^2}{4\hbar\Delta}|\vec{\mathcal{E}}|^2$

$$= -\frac{1}{4}\alpha|\vec{\mathcal{E}}|^2$$

as in the classical calculation (6)

Oscillator strength

$$f \equiv \frac{\alpha_q}{\alpha_c} = \left(\frac{|\langle e|\vec{d}|g\rangle|^2}{\hbar\Delta}\right)\left(\frac{2m\omega_0\Delta}{e^2}\right)$$

$$= |\langle e|z|g\rangle|^2\left(\frac{2m\omega_0}{\hbar}\right)$$

$$\boxed{f = \frac{|\langle e|z|g\rangle|^2}{(\Delta z)^2_{SHO}}} \quad \text{where} \quad (\Delta z)_{SHO} = \frac{\hbar}{2m\omega_0}$$

For a multi-level atom with resonances $\{\omega_i\}$

$$\alpha = \sum_i f(\omega_i)\,\alpha(\omega_i)$$

The oscillator strength satisfies the "sum rule"

$$\sum_i f(\omega_i) = Z \text{ (atomic \#)}$$

For Hydrogen and the alkalis, the majority of the oscillator strength lies in the ~~[illegible]~~ first $s \to p$ transition. Thus, if the perturbation is far from any resonance, this transition will dominate.

Problem 1: Light-Shift (multi-level atoms)

Arbitrary, vector monochromatic field $\vec{E}(\vec{x},t) = Re(\vec{E}(\vec{x})e^{-i\omega_L t})$

driving ground $\leftrightarrow$ excited manifolds

$$\{|g; J_g\rangle \longleftrightarrow |e; J_e\rangle\}$$

Light-shift operator on ground manifold

$$\hat{V}_{LS}(\vec{x}) = -\frac{1}{4}\vec{E}^*(\vec{x}) \cdot \hat{\overleftrightarrow{\alpha}} \cdot \vec{E}(\vec{x})$$

where $\hat{\overleftrightarrow{\alpha}} = -\frac{\hat{\vec{d}}_{ge}\hat{\vec{d}}_{eg}}{\hbar\Delta}$ polarizability tensor

$\hat{\vec{d}}_{eg} = \hat{\mathbb{P}}_e \hat{\vec{d}} \hat{\mathbb{P}}_g$ projectors

$\hat{\vec{d}}_{ge} = \hat{\vec{d}}_{eg}^{\dagger}$

(a) Explicit representation in basis of mag-sublevels:

$$\hat{\vec{d}}_{eg} = \hat{\mathbb{P}}_e \hat{\vec{d}} \hat{\mathbb{P}}_g = \sum_{M_e=-J_e}^{J_e} \sum_{M_g=-J_g}^{J_g} |e; J_e M_e\rangle\langle e; J_e M_e| \hat{\vec{d}} |g; J_g M_g\rangle\langle g; J_g M_g|$$

Spherical basis expansion: $\hat{\vec{d}} = \sum_q (-1)^q \vec{e}_{-q} \hat{d}_q = \sum_q \vec{e}_q^{\,*} \hat{d}_q$

$$\Rightarrow \hat{\vec{d}}_{eg} = \sum_{M_e, M_g} \vec{e}_q^{\,*} \langle e; J_e M_e| \hat{d}_q |g; J_g M_g\rangle \, |e; J_e M_e\rangle\langle g; J_g M_g|$$

Aside: W. E. T.

$$\langle e; J_e M_e | \hat{d}_q | g; J_g M_g \rangle = \langle e; J_e \| d \| g; J_g \rangle$$
$$\langle J_e M_e | 1\ q\ J_g M_g \rangle$$

Selection rule $M_e = M_g + q$

$$\Rightarrow \hat{\vec{d}}_{eg} = \sum_{M_g, q} \vec{e}_q^{\,*}\, C_{M_g}^{M_g+q}\, |e; J_e\, M_g+q\rangle\langle g; J_g\, M_g|$$
$$\langle e \| d \| g \rangle$$

Where I have used a short hand for the dipole C-G coef: $C_{m_g}^{m_g+q} = \langle J_e\, m_g+q | 1\ q\ J_g\ m_g \rangle$

Similarly $\hat{\vec{d}}_{ge} = \hat{d}_{eg}^{\dagger} = \sum_{M_g, q} \vec{e}_q\, C_{M_g}^{M_g+q}\, |g, J_g M_g\rangle\langle e, J_e\, M_g+q|$
$$\langle e \| d \| g \rangle^*$$

$$\Rightarrow \hat{d}_{ge}\, \hat{d}_{eg} = \sum_{M_g, M_g'} \sum_{q, q'} \vec{e}_{q'}\, \vec{e}_q^{\,*}\, C_{M_g'}^{M_g'+q'}\, C_{M_g}^{M_g+q}\, |\langle e \| d \| g \rangle|^2$$

Note:
Two sums must have two sets of indicies

$$|g, J_g M_g'\rangle \underbrace{\langle e; J_e\, M_g'+q' | e; J_e\, M_g+q \rangle}_{\delta_{M_g'+q',\, M_g+q}} \langle g\, J_g\, M_g|$$

$\Rightarrow$ Selection rule: $M_g' - M_g = q - q'$
(conservation of angular momentum)

$$\delta_{M_g' = M_g + q - q'}$$

(Next Page)

Thus,

$$\hat{d}_{ge}\,\hat{d}_{eg} = |\langle e\|d\|g\rangle|^2 \sum_{M_g} \sum_{q q'} \vec{e}_{q'}\,\vec{e}_q^{\,*}\, C^{M_g+q}_{M_g+q-q'}\, C^{M_g+q}_{M_g}\, |g; J_g, M_g+q-q'\rangle\langle J_g M_g|$$

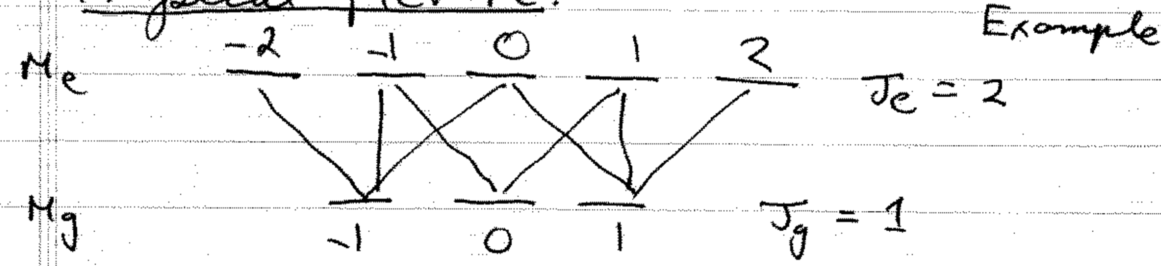
Putting this together, the polarizability tensor in a spherical basis representation:

$$\boxed{\begin{aligned}\hat{\overleftrightarrow{\alpha}} = \tilde{\alpha}\Big[&\sum_{M_g, q} |C^{M_g+q}_{M_g}|^2\, \vec{e}_q\, |g; J_g M_g\rangle\langle g; J_g M_g|\, \vec{e}_q^{\,*} \\ &+ \sum_{M_g, q\neq q'} C^{M_g+q}_{M_g+q-q'}\, C^{M_g+q}_{M_g}\, \vec{e}_{q'} |g; J_g M_g+q-q'\rangle\langle J_g M_g|\, \vec{e}_q^{\,*}\end{aligned}}$$

Here I explicitly write diagonal + off-diag terms

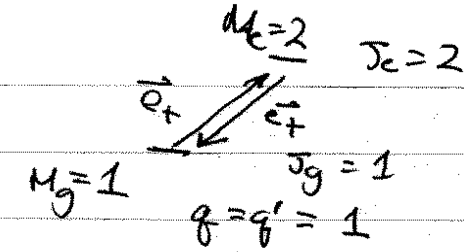
where $\tilde{\alpha} = -\dfrac{|\langle e; J_e\|d\|g, J_g\rangle|^2}{\hbar\Delta}$ (reduced matrix element)

<u>Physical picture:</u>



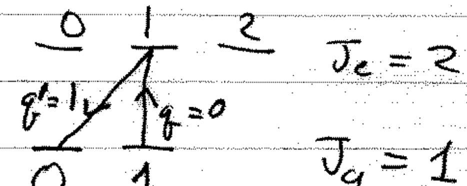
The shift can be thought of as resulting from virtual absorption and emission of photons. If the atom absorbs and reëmits a photon of helicity q ⇒ comes back to the same state ⇒ Oscillator strength $M_g \leftrightarrow M_e$.
If the atom absorbs photon helicity q and emits q', the deficit goes into atomic ang. mo... $M_g \Rightarrow M = M_g + q \rightarrow M_g = M_e - q'$

ex: Diagonal term



strength $\left| C_{M_g=1}^{M_e=2} \right|^2$

example: Off-diagonal term



Strength $\left(C_0^1 \, C_1^1 \right)$

(b) Plane polarized case: $\vec{E}(\vec{x}) = E_1 \vec{\varepsilon}_L e^{i\vec{k}\cdot\vec{x}}$

$$\Rightarrow \hat{V}_{LS} = -\frac{1}{4}|E_1|^2 \, \vec{\varepsilon}_L^{\,*} \cdot \hat{\overleftrightarrow{\alpha}} \cdot \vec{\varepsilon}_L$$

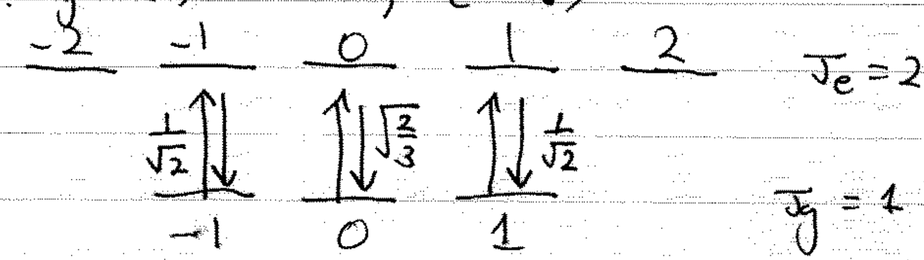
Case (i): Linear polarization along z $\quad \vec{\varepsilon}_L = \vec{e}_0$

$$\Rightarrow \hat{V}_{LS} = -\frac{1}{4}|E_1|^2 \, \vec{e}_0 \cdot \hat{\overleftrightarrow{\alpha}} \cdot \vec{e}_0$$

$$= \underbrace{-\frac{\tilde{\alpha}}{4}|E_1|^2}_{V_1} \sum_{M_g} |C_{M_g}^{M_g}|^2 \, |g; J_g M_g\rangle\langle g; J_g M_g|$$

$\Rightarrow$ Only diagonal terms (see picture above)

Case $|g; J_g = 1\rangle \longrightarrow |e; J_e = 2\rangle$



Only π-polarization

Clebsch-Gordon coeff $C_{M_g}^{M_g} = \langle 2 M_g | 1 0 \; 1 M_g \rangle$

⇒ For $\vec{\mathcal{E}}_L = \vec{e}_z$ $\quad |J_g = 1\rangle \to |J_e = 2\rangle$

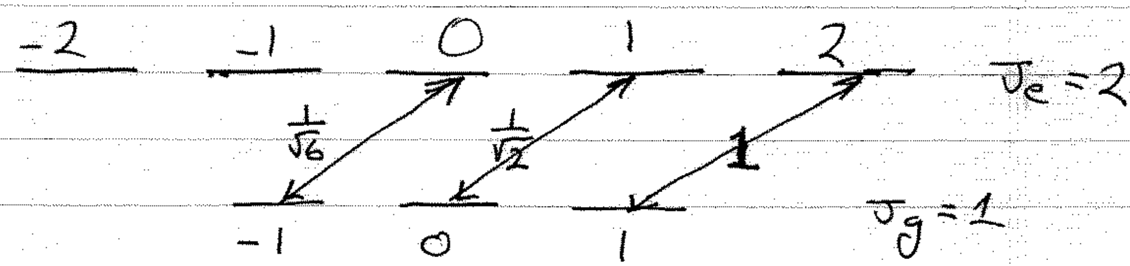
Eigenvectors: Magnetic sublevels $|g, J_g$

Eigenvalues: $\frac{1}{2} V_1$, $\frac{2}{3} V_1$

Sketch: $\Delta < 0 \Rightarrow V_1 < 0$

M_g $\quad -1 \quad 0 \quad 1$ $\quad$ Shifted levels

Case $\vec{\mathcal{E}}_L = \vec{e}_+$ $\quad |J_g = 1\rangle \to |J_e = 2\rangle$



Eigenvectors: $|J_g, M_g\rangle$

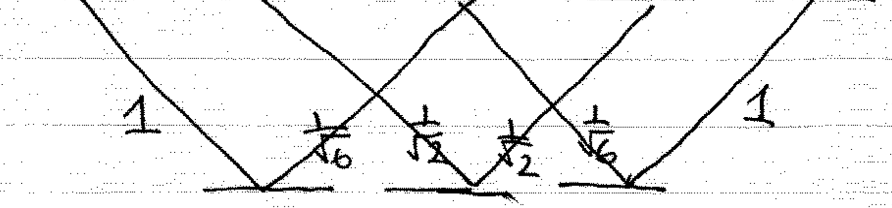
Eigenvalues: $M_g =$ [redacted] : $\frac{1}{6} V_1$, $M_g = 0$: $\frac{V_1}{2}$, $M_g = +1$: V_1

M_g $\quad -1 \quad 0 \quad 1$ $\quad$ ($V_1 < 0$ here)

Case $\vec{\mathcal{E}}_L = \vec{e}_x$ $\quad |J_g = 1\rangle \to |J_e = 2\rangle$

$\vec{e}_x = \dfrac{-\vec{e}_+ + \vec{e}_-}{\sqrt{2}}$; Not one of spherical basis

⇒ Not just diagonal elements



<u>Light Shift operator</u> for $\vec{\varepsilon}_L = \vec{e}_x = \frac{1}{\sqrt{2}}(-\vec{e}_{+} + \vec{e})$

$$\hat{V}_L(\vec{x}) = V_1 \Bigg[\sum_{M_g, q} |C_{M_g}^{M_g+q}|^2 (\vec{e}_x \cdot \vec{e}_q) |J_g M_g\rangle\langle J_g M_g| (\vec{e}_q^{\,*} \cdot \vec{e}_x)$$

$$+ \sum_{M_g, q, q'} C_{M_g+q-q'}^{M_g+q} \, C_{M_g}^{M_g+q} (\vec{e}_x \cdot \vec{e}_{q'}) |J_g, M_g+q-q'\rangle\langle J_g M_g| \vec{e}_q$$

$$= \frac{V_1}{2} \Bigg[\sum_{M_g} \left(|C_{M_g}^{M_g+1}|^2 + |C_{M_g}^{M_g-1}|^2 \right) |J_g M_g\rangle\langle J_g M_g|$$

$$- \sum_{M_g} \left(C_{M_g+2}^{M_g+1} \, C_{M_g}^{M_g+1} |J_g, M_g+2\rangle\langle J_g, M_g| + h.c. \right) \Bigg]$$

$$= \frac{V_1}{2} \left(|0\rangle\langle 0| + \frac{1}{12}\left(7|1\rangle\langle 1| + 7|-1\rangle\langle -1| - |1\rangle\langle -1| - |-1\rangle\langle 1| \right) \right)$$

here I have simplified notation $|M_g\rangle \equiv |J_g, M_g\rangle$

Eigenvalues are eigenvectors

In the basis $\{|0\rangle, |1\rangle, |-1\rangle\}$

$$\hat{V}_L \doteq \frac{V_1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{7}{12} & -\frac{1}{12} \\ 0 & -\frac{1}{12} & \frac{7}{12} \end{bmatrix}$$

Block diagonal. We must diagonize the 2×2 matrix

$$\begin{bmatrix} \frac{7}{12} & -\frac{1}{12} \\ -\frac{1}{12} & \frac{7}{12} \end{bmatrix} = \frac{7}{12}\mathbb{1} - \frac{1}{12}\sigma_x \Rightarrow \text{eigenvalues } \frac{7}{12} \mp \frac{1}{12} = \frac{6}{12}, \frac{8}{12}$$

eigenvectors $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{\varepsilon}_L = \vec{e}_x$

$J_g = 1 \to J_e = 2$ $\Rightarrow$

$$\boxed{\begin{array}{lll} \text{Eigenvector / eigenvalue} & & \\ |0\rangle : \frac{V_1}{2} & \frac{|1\rangle + |-1\rangle}{\sqrt{2}} : \frac{V_1}{2} & \frac{|1\rangle - |-1\rangle}{\sqrt{2}} : \frac{2}{3} V_1 \end{array}}$$

Thus we see that the eigenvalues for $\vec{\varepsilon}_L = \vec{e}_z$ and $\vec{\varepsilon}_L = \vec{e}_x$ are equal. this is as it must be. What direction we call "x" or "y" or "z" is irrelevant. the choice of "quantization axis" is arbitrary.

What about the eigenvectors for these two cases? For $\vec{\varepsilon}_L = \vec{e}_z$ with z-quantization we found eigenvectors $|M_z = 0\rangle$, $|M_z = \pm 1\rangle$ with eigenvalues

$$\hat{V}_L = \tfrac{2}{3}V_1 \ , \quad \tfrac{1}{2}V_1 \text{ (doubly degenerate)}$$

Thus for $\vec{\varepsilon}_L = \vec{e}_x$ with x-quantization we must have eigenvectors $|M_x = 0\rangle$, $|M_x = \pm 1\rangle$ with eigenvalues

$$\hat{V}_L = \tfrac{2}{3}V_1 \quad \tfrac{1}{2}V_1 \text{ (doubly degenerate)}$$

Now $|M_x\rangle = \hat{D}\,|M_z\rangle$ where $\hat{D}$ is a rotation matrix

For $J_g = 1$ we can explicitly calculate the rotation matrix, or use symmetry arguments via the spherical basis (see Phys 521, P.S. #6)

$$|1, M_x = 0\rangle = \frac{-1}{\sqrt{2}}\left(|1, M_z = +1\rangle - |1, M_z = -1\rangle\right)$$

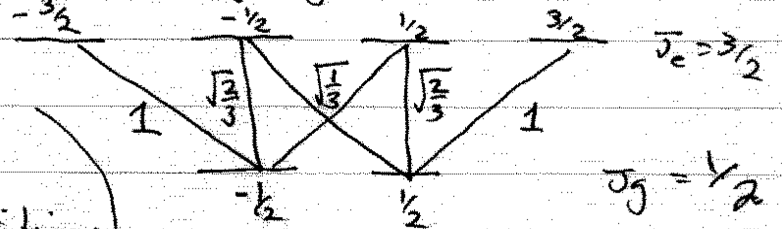
$$|1, M_x = \pm 1\rangle = \frac{-i}{\sqrt{2}}\left[\pm |1, M_z = 0\rangle + \frac{1}{\sqrt{2}}\left(|1, M_z = 1\rangle + |1, M_z = -1\rangle\right)\right]$$

thus $\hat{V}_L\,|M_x = 0\rangle = \tfrac{2}{3}V_1\,|M_x = 0\rangle$

$\hat{V}_L\,|M_x = \pm 1\rangle = \tfrac{1}{2}V_1\,|M_x = \pm 1\rangle$ ✓

So it all makes sense

Repeat for $|g, J_g = 1/2\rangle \rightarrow |e; J_e = 3/2\rangle$



(C-G coeff's for dipole transitions)

<u>Case (i)</u> $\vec{\mathcal{E}}_L = \vec{e}_z$

$\boxed{\text{Eigenvector } |J_g, M_g = \pm 1/2\rangle \quad \text{Eigenvalues } \tfrac{2}{3}V_1 \text{ (doubly degenerate)}}$

<u>Case (ii)</u> $\vec{\mathcal{E}}_L = \vec{e}_+$

$\boxed{\text{Eigenvectors } |J_g, M_g = \pm\tfrac{1}{2}\rangle \quad \text{Eigenvalues } |M_g = 1/2\rangle : V_1 , \quad |M_g = -1/2\rangle : \tfrac{1}{3}V_1}$

<u>Case (iii)</u> $\vec{\mathcal{E}}_L = \vec{e}_x = (-\vec{e}_+ + \vec{e}_-)/\sqrt{2}$

Unlike the $J_g = 1$ case, the light-shift operator is <u>diagonal</u> here since there are no $\Delta M_g = \pm 2$ coherences possible in the ground state

$$\hat{V}_{LS} = \frac{V_1}{2} \sum_{M_g} \left(|C_{M_g}^{M_g+1}|^2 + |C_{M_g}^{M_g-1}|^2 \right) |J_g M_g\rangle\langle J_g M_g|$$

$$= \frac{V_1}{2}\left[\left(1 + \tfrac{1}{3}\right)|\tfrac{1}{2}\rangle\langle\tfrac{1}{2}| + \left(\tfrac{1}{3} + 1\right)|-\tfrac{1}{2}\rangle\langle -\tfrac{1}{2}|\right)$$

$$= \frac{2}{3} V_1 \left(|\tfrac{1}{2}\rangle\langle\tfrac{1}{2}| + |-\tfrac{1}{2}\rangle\langle -\tfrac{1}{2}|\right)$$

$\Rightarrow$ Eigenvectors $|J_g, M_g = \pm 1/2\rangle$
Eigenvalues $\frac{2}{3} V_1$ (doubly ...)

Same as $\vec{\mathcal{E}}_L = \vec{e}_z$ as it must be

(c) The polarizability tensor can be written in terms of irreducible tensors

Let $\hat{T}_{ij} = d^i_{ge}\, d^j_{eg}$ outer product of two vectors

As discussed in class, any such Cartesian tensor can be expanded in terms of irreducible tensors

$$\hat{T}_{ij} = \hat{T}^{(0)}_{ij} + \hat{T}^{(1)}_{ij} + \hat{T}^{(2)}_{ij}$$

where $\hat{T}^{(0)}_{ij} = \mathrm{Trace}(\hat{T}_{ij})\, \frac{\delta_{ij}}{3} = \frac{1}{3}\, \vec{d}_{ge} \cdot \vec{d}_{eg}$

Antisymmetric: $\hat{T}^{(1)}_{ij} = \frac{d^i_{ge} d^j_{eg} - d^j_{ge} d^i_{eg}}{2} = \frac{1}{2} \epsilon_{ijk} \left(\vec{d}_{ge} \times \vec{d}_{eg}\right)_k$

Symmetric, Traceless: $\hat{T}^{(2)}_{ij} = \frac{d^i_{ge} d^j_{eg} + d^j_{ge} d^i_{eg}}{2} - \frac{1}{3} \vec{d}_{ge} \cdot \vec{d}_{eg}$

Thus with $\hat{\alpha}_{ij} = -\frac{1}{\hbar\Delta} \hat{T}_{ij}$

we arrive at the desired expansion

$$\hat{V}_{LS} = -\frac{1}{4} \vec{E}^*(\vec{x}) \cdot \hat{\alpha}\, \vec{E}(\vec{x}) = -\frac{1}{4} E_i^* E_j\, \hat{\alpha}_{ij}$$

$$= -\frac{1}{4}\left(\hat{\alpha}^{(0)} |\vec{E}(\vec{x})|^2 + \hat{\alpha}^{(1)} \cdot (\vec{E}^* \times \vec{E}) + \vec{E}^* \cdot \hat{\alpha}^{(2)} \cdot \vec{E}\right)$$

$$\hat{\alpha}^{(K)}_{ij} = -\frac{1}{\hbar\Delta} \hat{T}^{(K)}_{ij}$$

(d) For the particular case $|g; J_g = 1/2\rangle \to |e; J_e = 3/2\rangle$

$\hat{V}_{LS}$ acts on the ground manifold $\{|J_g, 1/2\rangle, |J_g, -1/2\rangle\}$

The rank-2 part $\hat{V}_{LS}^{(2)} = -\frac{1}{4} E_i^* E_j \hat{\alpha}_{ij}^{(2)}$

has matrix elements $\langle J_g = 1/2, M_g' | \hat{V}_{LS}^{(2)} | J_g = 1/2, M_g\rangle$

$$= \langle \tfrac{1}{2} \| \hat{V}_{LS}^{(2)} \| \tfrac{1}{2}\rangle \underbrace{\langle \tfrac{1}{2} M_g' | 2 q \; \tfrac{1}{2} M_g\rangle}$$

this C-G coeff vanishes
by the triangle inequality

Thus,
$$\hat{V}_{LS} = \underbrace{-\frac{1}{4} |\vec{E}(x)|^2 \alpha^{(0)}}_{\text{Scalar}} \underbrace{- \frac{1}{4} (\vec{E}^* \times \vec{E}) \cdot \hat{\alpha}^{(1)}}_{\text{Vector}}$$

Acting on spin-1/2 the scalar part must be proportional to the identity and the vector part to $\hat{\vec{\sigma}}$ since any operator acting on the 2D ~~Hilbert~~ Hilbert space is of this form

Thus,
$$\boxed{\begin{aligned} \hat{V}_{LS} &= V_0(\vec{x}) \hat{\mathbb{1}} + \vec{B}_{eff}(\vec{x}) \cdot \hat{\vec{\sigma}} \\ \text{Where } V_0(\vec{x}) &= \tfrac{1}{2} \mathrm{Tr}(\hat{V}_{LS}) \\ \vec{B}_{eff} &= \tfrac{1}{2} \mathrm{Tr}(\hat{V}_{LS} \hat{\vec{\sigma}}) \end{aligned}}$$