Problem Set #17: Solutions

The beam splitter and other linear transformations

\[ E_a^{(out)} \quad E_b^{(out)} \]
\[ E_a^{(in)} \quad \quad \quad E_b^{(in)} \]
\[ \begin{bmatrix}
E_a^{(out)} \\
E_b^{(out)}
\end{bmatrix} =
\begin{bmatrix}
t & r \\
1 & t
\end{bmatrix}
\begin{bmatrix}
E_a^{(in)} \\
E_b^{(in)}
\end{bmatrix} \]

"S-matrix" \( S \)

(a) Unitarity of the S-matrix: \( S^\dagger S = 1 \)

\[ SS^\dagger = \begin{bmatrix}
t & r \\
r & t
\end{bmatrix}
\begin{bmatrix}
t^* & r^* \\
r^* & t^*
\end{bmatrix} = \begin{bmatrix}
t|t|^2 + |r|^2 & tr^* + t^*r \\
tr^* + t^*r & t^*|t|^2 + |r|^2
\end{bmatrix} \]

\[ \Rightarrow |t|^2 + |r|^2 = 1 \quad \quad \quad \text{Re}(t^*r) = 0 \]

let \( T = |t|^2 \)

\[ \Rightarrow t = \sqrt{T} e^{i\phi_t} \quad r = i \sqrt{1-T} e^{i\phi_t} \]

\( \phi_t \) depends on details on beam splitter

For \( \phi_t = 0 \)

\[ E_a^{(out)} = \sqrt{T} E_a^{(in)} + \sqrt{1-T} E_b^{(in)} \]
\[ E_b^{(out)} = \sqrt{T} E_b^{(in)} + \sqrt{1-T} E_a^{(in)} \]
(b) Quantized mode: \( E_a \rightarrow \hat{a} \quad E_b \rightarrow \hat{b} \)

Suppose no field is injected into port "b".

Classically \( E_a^{(out)} = JT E_a^{(in)} \)

Quantum analogy \( \hat{a}^{(out)} = JT \hat{a}^{(in)} \quad ? \)

No \( \{\hat{a}^{(out)}, \hat{a}^{(out)\dagger}\} = T \{\hat{a}^{(in)}, \hat{a}^{(in)\dagger}\} = T \leq 1 \)

(c) So the uncertainty principle would be violated.

The problem is that we allowed attenuation of vacuum fluctuations. Formally we violated unitarity in the transformation between input and output.

\[ \hat{a}^{(out)} = S^+ \hat{a}^{(in)} = \sqrt{T} \hat{a}^{(in)} + i \sqrt{1-T} \hat{\mathbf{1}} \]

\[ \Rightarrow \{\hat{a}^{(out)}, \hat{a}^{(out)\dagger}\} = \hbar c^2 \{\hat{a}^{(in)}, \hat{a}^{(in)\dagger}\} + 2Tr \{\hat{a}^{(in)}\hat{a}^{(in)\dagger}\} \]

One way to interpret this is that although we do not input a signal into port \( b \), vacuum fluctuations always enter that port.
(a) Input single photon into mode-a and nothing in mode-b:

\[ \text{Input: } \hat{a}^\dagger |0\rangle_b \]

\[ \Rightarrow |\Phi_{\text{in}}\rangle = |a\rangle_a \otimes |0\rangle_b = \hat{a}^{(\text{in}) \dagger} |0,0\rangle \]

\[ \hat{a}^{(\text{out})} |\Phi_{\text{in}}\rangle = \hat{a}^{(\text{out}) \dagger} |0,0\rangle \]

\[ = (t \hat{a}^{(\text{in}) \dagger} + r \hat{b}^{(\text{in}) \dagger}) |0,0\rangle \]

\[ = t |1,0\rangle + r |0,1\rangle \]

\[ |\Phi_{\text{out}}\rangle = t |1\rangle_a \otimes |0\rangle_b + r |0\rangle_a \otimes |1\rangle_b \]

(e) Now suppose we inject a coherent state:

\[ \hat{a}^{(\text{in}) \dagger} (\alpha) = \exp \{ \alpha \hat{a}^{(\text{in}) \dagger} + \alpha^* \hat{a}^{(\text{in})} \} \]

\[ \Rightarrow |\Phi_{\text{in}}\rangle = |\alpha\rangle_a \otimes |0\rangle_b \]

\[ = D_{a}^{(\alpha)} |0,0\rangle \]

\[ D_{a}^{(\alpha)} (\alpha) = \exp \{ \alpha \hat{a}^{(\text{in}) \dagger} + \alpha^* \hat{a}^{(\text{in})} \} \]

\[ \Rightarrow |\Phi_{\text{out}}\rangle = \hat{a}^{(\text{out}) \dagger} |\Phi_{\text{in}}\rangle = \hat{a}^{(\text{out}) \dagger} D_{a}^{(\alpha)} |0,0\rangle \]

\[ = \exp \{ \alpha \hat{a}^{(\text{in}) \dagger} + \alpha^* \hat{a}^{(\text{in})} \} D_{a}^{(\alpha)} |0,0\rangle \]

\[ = \exp \{ \alpha \hat{a}^{(\text{out}) \dagger} + \alpha^* \hat{a}^{(\text{out})} \} + \exp \{ \alpha \hat{b}^{(\text{in}) \dagger} + \alpha^* \hat{b}^{(\text{in})} \} \]

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since $\hat{A}^{(m)}$ and $\hat{B}^{(m)}$ modes commute

$$|\psi_{\text{out}}\rangle = \hat{A}_b^{(m)} \hat{B}_a^{(m)} |10,0\rangle$$

$$U |\psi_{\text{out}}\rangle = |t\alpha\rangle_a \otimes |r\alpha\rangle_b$$

This is the classically expected result

![Diagram](https://via.placeholder.com/150)

This is in contrast to the state

$$\frac{1}{\sqrt{2}} (|\alpha\rangle_a \otimes |10\rangle_b + |r\rangle_a \otimes |1\rangle_b)$$

which is a "Schrödinger cat" state. This state describes a superposition of two macroscopic outcomes: the entire beam is transmitted (with probability $|t|^{2}$) or the entire beam is reflected (with probability $|r|^{2}$). This is a very nonclassical transformation, not accomplished by the linear beam splitter. The coherent state is basically a many photon copy of the single photon state. Each photon acts independently and randomly takes the transmitted or reflected path; the Poisson statistics are preserved.
(f) Model of an imperfect detector

The beam splitter acts to model the fact that a photon present in the field will only be detected with finite probability $\eta$.

\[ a_{\text{in}} \xrightarrow{\text{splitter}} a_{\text{out}} = \sqrt{\eta} a_{\text{in}} + i \sqrt{1-\eta} b_{\text{in}} \]

The input field (consider pure state)

\[ |\Psi_{\text{in}}\rangle = \sum_n c_n |n_a\rangle \otimes |n_b\rangle = \sum_n c_n (\hat{a}_{\text{in}}^+)^n |0_a\rangle \otimes |0_b\rangle \]

The output field

\[ |\Psi_{\text{out}}\rangle = \sum_n c_n \left( \sqrt{\eta} \hat{a}_{\text{in}}^+ + i \sqrt{1-\eta} \hat{b}_{\text{in}}^+ \right)^n |0_a\rangle \otimes |0_b\rangle \]

\[ = \sum_{n,m} c_n \frac{(\eta)^m}{\sqrt{n!}} (\eta_m) (1 - \eta)^{n-m} \hat{a}_{\text{in}}^m \hat{b}_{\text{in}}^{n-m} |n_a\rangle \otimes |n_b\rangle \]

where \( \begin{pmatrix} n \\ m \end{pmatrix} = \frac{n!}{m! (n-m)!} \) is the binomial coefficient.

\[ |\Psi_{\text{out}}\rangle = \sum_{n,m} c_n (i)^{n-m} \sqrt{(\eta_m) (1 - \eta)^{n-m}} |m_a\rangle \otimes |n-m_b\rangle \]

where I used $|m_a\rangle = \frac{\hat{a}_{\text{in}}^m}{\sqrt{m!}} |10\rangle_a$, 

\[ |n-m_b\rangle = \frac{(\hat{b}_{\text{in}}^*)^{n-m}}{\sqrt{(n-m)!}} |10\rangle_b \]
We seek the probability of detecting \( m \) photons in the a-channel, irrespective of the number of photons in the b-channel.

\[
P_m = \sum_{n_0} |<m_a, n_0| Y^{out} Y|^2
\]

\[
= \sum_{n_0} |c_{n_0}|^2 \left( \frac{n}{m} \right) \eta^m (1-\eta)^{n-m}
\]

\[
= \sum_n P_n \left( \frac{n}{m} \right) \eta^m (1-\eta)^{n-m}
\]

Bernoulli distribution

This expression has a simple interpretation. 
\( \eta \) is the probability of detecting one photon. 
Assuming statistically independent detections, 
given \( n \) photons the probability of detection \( m \) events is \((P(1\text{ photon}))^m \times \text{Probability of not detecting } n-m \)

\[
\eta^m (1-\eta)^{n-m}
\]

There are \( \binom{n}{m} \) different ways of choosing \( m \) out of \( n \) photons. The total probability 
given \( n \) photons \( P(m|n) = \binom{n}{m} \eta^m (1-\eta)^{n-m} \)

\[
\Rightarrow P_m = \sum_n P_n P(m|n) \text{ as above.}
\]

Note: Generally the photo-electron statistics will not be a faithful reconstruction of the photon statistics. The exception is for classical light.
(g) Linear optics $\Rightarrow$ linear transformation of modes

$$E_{k}^{\text{(out)}} = \sum_{k'} U_{k'k}^* E_{k'}^{\text{(in)}}$$

Quantum mechanically, $\hat{a}_{k}^{(\text{out})} = \hat{S} \hat{a}_{k}^{(\text{in})} \hat{S}^\dagger = \sum_{k'} U_{k'k} \hat{a}_{k'}^{(\text{in})}$

Suppose we start in an arbitrary multimode coherent state

$$\left| 12^{\text{in}} \right> = D^{(\text{in})}(\xi \exp i \vec{k}) | 0 \rangle = \prod_{k} D^{(\text{in})}(\xi) | 0 \rangle$$

$$= \prod_{k} \exp \left( \frac{i}{\hbar} \vec{\alpha}_k^{\text{(in)}} - \vec{\alpha}_{k}^{\ast} \hat{a}_{k}^{(\text{in})} \right) = \exp \left[ \sum_{k} \left( \frac{i}{\hbar} \vec{\alpha}_k^{\text{(in)}} - \vec{\alpha}_{k}^{\ast} \hat{a}_{k}^{(\text{in})} \right) \right] | 0 \rangle$$

$$\left| 12^{\text{out}} \right> = \hat{S} \left| 12^{\text{in}} \right> = \exp \left[ \sum_{k} \left( \vec{\alpha}_k^{\text{(out)}} - \vec{\alpha}_{k}^{\ast} \hat{a}_{k}^{(\text{out})} \right) \right] | 0 \rangle$$

$$= \exp \left[ \sum_{k} \left( \vec{\alpha}_k^{\ast} \hat{a}_{k}^{(\text{in})} - \vec{\alpha}_{k}^{\ast} \hat{a}_{k}^{(\text{in})} \right) \right] | 0 \rangle$$

where I have used $U_{k'k}^* = U_{k'k}$ for unitary matrix

$$\Rightarrow \left| 12^{\text{out}} \right> = \exp \left[ \sum_{k} \left( \vec{\alpha}_k^{\ast} \hat{a}_{k}^{(\text{in})} - \vec{\alpha}_{k}^{\ast} \hat{a}_{k}^{(\text{in})} \right) \right] | 0 \rangle$$

where $\vec{\alpha}_k^{\ast} = \sum_k U_{k'k} \vec{\alpha}_{k'}^{\ast}$

$$\Rightarrow \left| 12^{\text{out}} \right> = D(\xi \exp i \vec{k}) | 0 \rangle$$
(b) A non-classical input leads to non-classical phenomena, even for linear transformations.

Suppose $|12^\text{in}\rangle = |1\rangle_a \otimes |1\rangle_b$, two "mode-matched" single photons incident simultaneously on a beam splitter:

$$\rightarrow |12^\text{in}\rangle = a^+ \otimes b^+ \rightarrow |0\rangle$$

\[
|12^\text{out}\rangle = \frac{1}{\sqrt{2}} (a^+ b^+ - i a^+ b^+ (i)^2 a^+ b^+) |10\rangle
\]

Overall phase:

$$= \frac{\pi}{2} \left( a^2 + b^2 \right) |10\rangle + \frac{1}{\sqrt{2}} (a^+ b^+ + a^+ b^+) |0\rangle$$

Since $[a^+ b^+] = 0$

$$\Rightarrow |12^\text{out}\rangle = \frac{1}{\sqrt{2}} \left( |1\rangle_a |0\rangle_b + |0\rangle_a |1\rangle_b \right)$$

Thus both photons go off together.

We see that there is destructive interference for the two processes below:

\[ (i) \quad a^+ \quad b^+ \quad a^+ \quad b^+ \]

Both transmitted, both reflected. Each picks up $\frac{\pi}{2}$ phase shift causing destructive interference.