**Problem 1: Lorentz Classical Model of Absorption and Emission (15 points)**

Suppose we were to model an atom as an electron on a spring - i.e. a damped simple harmonic oscillator of mass $m$, with resonance frequency $\omega_0$, and damping constant $\Gamma$. Consider driving the oscillator with a monochromatic plane wave, of frequency $\omega_L$.

(a) Show that rate at which the dipole absorbs energy from the field, given by the rate at which the field does work on the charge averaged over one period, is

$$\frac{dW_{abs}}{dt} = \frac{\pi e^2 |E|^2}{4m} g(\omega_L),$$

where

$$g(\omega) = \frac{\Gamma / (2\pi)}{(\omega - \omega_{r})^2 + \Gamma^2 / 4}$$

is the line shape.

Assume near resonance so that $\Delta = \omega_L - \omega_0 << \omega_L, \omega_0$.

(b) The absorption cross section, $\sigma_{abs}$, is defined as the rate at which energy is absorbed by an atom, divided by the flux, $\Phi$, of photons incident on the atom, $\Phi \equiv I / h\omega_L$ (i.e. the rate of photons incident on the atom per unit area), where

$$I = \frac{c}{8\pi} |E_0|^2$$

is the incident intensity (CGS units). Show that the classical model of absorption gives,

$$\sigma_{classical} = \frac{2\pi^2 e^2}{mc} g(\omega_L),$$

**Evaluate** this on-resonance, for a the parameters associated with Na, where the excitation wavelength is 589 nm and the linewidth (Full width at half-maximum) is 10 MHz.

The ratio of the integrated cross section an atomic transition and that given by the classical model to the quantum mechanical expression with equal resonance frequency and line width is known as the oscillator strength of the transition.
(c) From standard texts we have \( \sigma_{\text{quantum}} = 4\pi^2 \frac{e^2}{\hbar c} |\langle e|\mathbf{x}|g\rangle|^2 \omega_L g(\omega_L) \), where \( \langle e|\mathbf{x}|g\rangle \) is the matrix element of the electron position relative to the nucleus for the resonant transition. Show that on resonance,

\[
f = \frac{\sigma_{\text{quantum}}}{\sigma_{\text{classical}}} = \frac{2m\omega_0}{\hbar} |\langle e|\mathbf{x}|g\rangle|^2.
\]

Note that \( \hbar / 2m\omega_0 \) is the square of the characteristic length scale of a quantum simple harmonic oscillator. Thus, the oscillator strength measures the ratio of dipole oscillation amplitude for a two level atom as compared to a simple harmonic oscillator.

Let us now assume that our spring has no intrinsic damping associated with it. Consider the scattering of an electromagnetic wave by this oscillating charge. As the charge radiates, the electromagnetic field will carry away energy. This energy must come from the kinetic energy of the accelerating charge. Thus the very act of radiating should "damp" the motion of the charge. This is known as radiation reaction, and will determine a classical decay rate \( \Gamma_{\text{class}} \) for the oscillator. In steady state the power radiated by the charge (given by the classical Larmor formula) is equal to the power absorbed.

(d) Assume that the oscillator is damped as \( \Gamma_{\text{class}} \), and show that

\[
P_{\text{abs}} = P_{\text{radiated}} \Rightarrow \Gamma_{\text{class}} = \frac{2}{3} \frac{e^2}{mc^3} \omega^2 = \frac{2}{3} kr_c \omega, \text{where } r_c \text{ is the classical electron radius.}
\]

(e) Show that the quantum mechanical decay rate is related to the classical formula by

\[
\Gamma_{\text{quantum}} = f \Gamma_{\text{class}}, \text{ where } f \text{ is the oscillator strength.}
\]

**Problem 2: Radiation reaction and decay of the quantum oscillator** (15 points)
Radiation reaction can be shown to be lead to the decay of the quantum mechanical oscillator as well if we work in the Heisenberg picture. Start with the total Hamiltonian for a two level atom interacting with the quantized field, as discussed in class,

\[
\hat{H} = \frac{1}{2} \hbar \omega_{eg} \hat{\sigma}_z + \sum_{k\lambda} \hbar \omega_k \hat{a}^\dagger_{k\lambda} \hat{a}_{k\lambda} - \sum_{k\lambda} \hbar \left( g_{k\lambda} \hat{\sigma}_z \hat{a}_{k\lambda} + g_{k\lambda}^* \hat{a}^\dagger_{k\lambda} \hat{\sigma}_z \right),
\]

where \( g_{k\lambda} = i \sqrt{\frac{2\pi \hbar \omega_k}{V}} \hat{e}_{k\lambda} \cdot \mathbf{d}_{eg} \).
Note: We have expressed the interaction operator in "normally ordered" form, so that
annihilation operators are to the right and creation operators are to the left. We have
complete freedom to do this since field and atomic operators commute at equal times.

(a) Show that the Heisenberg equations of motion are:

\[
\frac{d}{dt} \hat{a}_{k\lambda} = -i\omega_{\lambda}\hat{a}_{k\lambda} + ig_{k\lambda}\hat{\sigma}_- \\
\frac{d}{dt} \hat{\sigma}_- = -i\omega_{\lambda}\hat{\sigma}_- - i\sum_{k,\lambda} g_{k\lambda}\hat{\sigma}_+ \hat{a}_{k\lambda} \\
\frac{d}{dt} \hat{\sigma}_+ = 2i\sum_{k,\lambda} \left( g_{k\lambda}\hat{\sigma}_+ \hat{a}_{k\lambda} - g^{*}_{k\lambda}\hat{a}_{k\lambda}^{\dagger}\hat{\sigma}_- \right)
\]

(b) The first equation is linear in the operators, and so can be formally integrated. Show
that

\[
\hat{a}_{k\lambda}(t) = \hat{a}_{k\lambda}(0)e^{-i\omega_{\lambda}t} + ig_{k\lambda}\int_0^t \hat{\sigma}_-(t')e^{-i\omega_{\lambda}(t-t')} dt' \\
\hat{\sigma}_+(t) = \hat{\sigma}_+(0)e^{-i\omega_{\lambda}t} + (t)
\]

is a solution. The first term \(\hat{a}_{k\lambda}^{\text{vac}}(t)\) is known as the vacuum field operator and \(\hat{a}_{k\lambda}^{\text{source}}(t)\) is
known as the source component due to dipole radiation by the atom.

(c) Show that, in general, given \([\hat{a}_{k\lambda}(t),\hat{a}_{k'\lambda'}^{\dagger}(t)] = \delta_{\lambda\lambda'}\delta_{k,k'}\), unitary evolution implies
\([\hat{a}_{k\lambda}(0),\hat{a}_{k'\lambda}^{\dagger}(0)] = \delta_{\lambda\lambda'}\delta_{k,k'}\). Show that the source part alone does not satisfy these relations.

(d) Plug the solution (b) back into the equation for \(\hat{\sigma}_+\). Take the Heisenberg state of the
system (initial state in the Schrodinger picture) to be \(|\Psi\rangle = |\psi\rangle_{\text{atom}} \otimes |0\rangle_{\text{field}}\), i.e., arbitrary
state of the atom plus field in the vacuum. Shown that expectation value satisfies

\[
\frac{d}{dt}\langle \hat{\sigma}_+ \rangle = -2\sum_{k\lambda} |g_{k\lambda}|^2 \int_0^t \langle \hat{\sigma}_-(t)\hat{\sigma}_+(t') \rangle e^{-i\omega_{\lambda}(t-t')} dt' + c.c
\]

(e) Now let us make the Markov approximation. Assume that \(\hat{\sigma}_-(t) = \hat{\Sigma}_-(t)e^{-i\omega_{\text{eg}}t}\), where
\(\hat{\Sigma}_-(t)\) is a slowly varying operator on the scale of \(\omega_{\text{eg}}\). Under this assumption
\(\langle \hat{\Sigma}_+(t)\hat{\Sigma}_-(t') \rangle = \langle \hat{\Sigma}_+(t)\hat{\Sigma}_-(t) \rangle = \langle \hat{\sigma}_+(t)\hat{\sigma}_-(t) \rangle\). Use this approximation to show,
\[
\frac{d}{dt}\langle \hat{\sigma}_z \rangle = -\Gamma (1 + \langle \hat{\sigma}_z \rangle),
\]
where \( \Gamma = \sum_{k\ell} 2\pi |g_{k\ell}| \int \delta (\omega_k - \omega_{e\ell}) \) is the Einstein A coefficient!

This is the expected decay! Note that vacuum fluctuations play no role in determining the rate of emission. They just initiate the process if the initial dipole moment is zero.

**Problem 1: Boson Algebra** (10 Points)


(a) Gaussian integrals in phase-space are used all the time. Show that
\[
\int \frac{d^2 \beta}{\pi} e^{-A|\beta|^2} e^{i\beta \cdot \alpha} = \frac{1}{A} e^{-|\alpha|^2 / A}.
\]

(b) Prove the completeness integral for coherent states
\[
\int \frac{d^2 \alpha}{\pi} |\alpha\rangle \langle \alpha| = 1 \text{ (Hint: Expand in number states).}
\]

(c) The “quadrature” operators in optics are the analogs of \( Q \) and \( P \), \( \hat{a} = \hat{X}_1 + i\hat{X}_2 \). Show
\[
\hat{U}(\theta) \hat{X}_1 \hat{U}^\dagger(\theta) = \cos \theta \hat{X}_1 + \sin \theta \hat{X}_2,
\]
\[
\hat{U}(\theta) \hat{X}_2 \hat{U}^\dagger(\theta) = \cos \theta \hat{X}_2 - \sin \theta \hat{X}_1,
\]
where \( \hat{U}(\theta) = e^{-i\theta \hat{a}^\dagger \hat{a}} \).

Interpret in phase space.

(d) Prove the group property of the displacement operator
\[
\hat{D}(\alpha) \hat{D}(\beta) = \hat{D}(\alpha + \beta) \exp \left\{ i \text{Im} (\alpha \beta^\dagger) \right\}
\]

(e) Show that the displacement operators has the following matrix elements

Vacuum: \( \langle 0 | \hat{D}(\alpha) | 0 \rangle = e^{-|\alpha|^2 / 2} \)

Coherent states: \( \langle \alpha | \hat{D}(\alpha) | \alpha \rangle = e^{-|\alpha|^2 / 2} e^{i \text{Im} (\alpha \alpha^\dagger - \alpha^\dagger \alpha)} \)

Fock states: \( \langle n | \hat{D}(\alpha) | n \rangle = e^{-|\alpha|^2 / 2} L_n (|\alpha|^2) \), where \( L_n \) is the Laguerre polynomial of order \( n \)