Problem 1: Coupled Simple Harmonic Oscillators and Entangled States (15 Points)

Consider two simple Harmonic oscillators, each with a natural frequency \( \omega \), linearly coupled together. The Hamiltonian describing such a system can be written:

\[
\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}
\]

\[
\hat{H}_0 = \hbar \omega \hat{a}^\dagger \hat{a} + \hbar \omega \hat{b}^\dagger \hat{b},
\]  

\[
\hat{H}_{\text{int}} = \hbar \kappa (\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}).
\]

where \( \hat{a} \) and \( \hat{b} \) are the annihilation operators for the two oscillators satisfying the commutation relations,

\[
[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{b}, \hat{b}^\dagger] = 1, \quad [\hat{a}, \hat{b}] = [\hat{a}, \hat{b}^\dagger] = 0,
\]

and \( \kappa \) is the coupling constant.

Recall the "interaction picture", defined by performing the unitary transformation on the states and operators in the Schrödinger picture,

\[
|\psi^{(I)}\rangle = \hat{U}_0|\psi^{(S)}\rangle, \quad \hat{A}^{(I)} = \hat{U}_0^\dagger \hat{A}^{(S)} \hat{U}_0, \quad \text{with} \quad \hat{U}_0 = \exp\left(\frac{-i}{\hbar} \hat{H}_0 t\right).
\]

(a) Show that in the interaction picture, the state vector evolves according to

\[
\frac{i\hbar}{\partial t} |\psi^{(I)}\rangle = \hat{H}_{\text{int}}^{(I)} |\psi^{(I)}\rangle.
\]

and for the Hamiltonian above, \( \hat{H}_{\text{int}}^{(I)} \) is time independent so that, the state vector in the Schrödinger picture

\[
|\psi(t)\rangle^{(S)} = \hat{U}_0 \exp\left(\frac{-i}{\hbar} \hat{H}_{\text{int}}^{(I)} t\right) |\psi(0)\rangle^{(I)} \quad \text{(Note that} \quad |\psi(0)\rangle^{(S)} = |\psi(0)\rangle^{(I)} )
\]

(b) Show that,

\[
\exp\left(\frac{-i}{\hbar} \hat{H}_{\text{int}}^{(I)} t\right) \hat{a} \exp\left(\frac{+i}{\hbar} \hat{H}_{\text{int}}^{(I)} t\right) = \hat{a} \cos(\kappa t) - i \hat{b} \sin(\kappa t),
\]

where \( \hat{a}, \hat{b} \) are the Schrödinger picture operators.
(c) Suppose that the initial state (in either the Schrödinger or Interaction picture) has oscillator "a" in the first excited state, and oscillator "b" in the ground state:

\[ |\psi(0)\rangle^{(S)} = |1\rangle_a \otimes |0\rangle_b = a^\dagger |0\rangle_a \otimes |0\rangle_b. \]

Show that the state vector for the total system at some later time \( t > 0 \) is,

\[ |\psi(t)\rangle^{(S)} = e^{-i\omega t} \cos(\kappa t) |1\rangle_a \otimes |0\rangle_b + i e^{-i\omega t} \sin(\kappa t) |0\rangle_a \otimes |1\rangle_b. \]

Explain what this result means physically.

(d) Find the density matrix for the total system as a function of time. Is it a pure state?

(e) Show that the reduced density matrix for the state of oscillator "a" alone is,

\[ \rho^{(a)}_{\text{reduced}} = \cos^2(\kappa t) |1\rangle_a \langle 1| + \sin^2(\kappa t) |0\rangle_a \langle 0|. \] Is it normalized.

(f) Show that \( \rho^{(a)}_{\text{reduced}} \) oscillates between a pure state and a mixed state with a frequency, \( \Omega = 2\kappa \). Explain this result.

**Problem 2:** State preparation in cavity QED. (10 points)

In the Jaynes-Cummings system, the atom and field become entangled. By measuring the state of the atom at some time, we gain information about the state of the field. Consider the following geometry,

A stream of atoms, prepared in a very long lived excited state, are sent through a very high-Q cavity interacting with the mode for a time \( \tau \). After the atom emerges, it is measured and determined to be in either the excited or ground state. The cavity is initially prepared in the vacuum before any atoms enter.

(a) Show that after \( m \) atoms interact and are measured, the probability of having \( n \) photons in the cavity satisfies the recursion relation

\[ P_n(m) = \cos^2\left(g \tau \sqrt{n+1}\right) P_n(m-1) + \sin^2\left(g \tau \sqrt{n}\right) P_{n-1}(m-1). \]
(b) Solve numerically, and plot as a function of $n$ for $m=5, 25, 50, 100$. Take $g\tau=0.4$. Comment.